COS 488 Week 6: Q3

Dylan Mavrides

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Solve the recurrence for p_N given in the proof of Theorem 8.9, to within the oscillating term.

$$p_N = \frac{1}{2^N}\sum_k \binom{N}{k} p_k$$

for $N > 1$ with $p_0 = 0$ and $p_1 = 1$.
This gives us the EGF $P(z) = e^{z/2}P(z/2) + z/2$ Where we add the z/2 term to account for the initial conditions. We see that expanding it gives us:

$$P(z) = e^{z/2}p(z/2)+z/2 = e^{z/2}(e^{z/4}p(z/4)+z/4)+z/2 = e^{z/2}(e^{z/4}(e^{z/8}p(z/8)+z/8)+z/4)+z/2 = ...$$

$$= z\sum_{i\geq 0} e^{z(1-2^{-i})}/2^{i+1}$$

which if we expand, gives the coefficients:

$$p_n = N![z^N]P(z) = N\sum_{i\geq 0}(1-1/2^i)^{N-1}/2^{i+1} \sim N\sum_{i\geq 0} e^{-N/2^i}/2^{i+1}$$

which we now use to isolate our periodic terms:

$$= \sum_{0\leq i<\lfloor \lg N\rfloor} e^{-N/2^i}/2^{i+1} + \sum_{i\geq \lfloor \lg N\rfloor} e^{-N/2^i}/2^{i+1}$$

$$= \sum_{i<\lfloor \lg N\rfloor} e^{-N/2^i}/2^{i+1} + \sum_{i\geq \lfloor \lg N\rfloor} e^{-N/2^i}/2^{i+1} + O(e^{-N})$$

$$= \sum_{i<0} e^{-N/2^{i+\lfloor \lg N\rfloor}}/2^{i+\lfloor \lg N\rfloor+1} + \sum_{i\geq 0} e^{-N/2^{i+\lfloor \lg N\rfloor}}/2^{i+\lfloor \lg N\rfloor+1} + O(e^{-N})$$

$$= \sum_{i<0} e^{-2^{\{\lg N\}-i}}/(N2^{i-\{\lg N\}+1}) + \sum_{i\geq 0} e^{-2^{\{\lg N\}-i}}/(N2^{i-\{\lg N\}+1}) + O(e^{-N})$$