Homework 6: Exercise 8.57

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First, we will make the recurrence valid for all N:

$$p_N = \frac{1}{2^N}\sum_k \binom{N}{k} p_k + \frac{1}{2}\delta_1$$

Multiplying both sides by z^N summing over N gives:

$$\begin{aligned}
p(z) &= \sum_{N=0}^{\infty}\sum_{k=0}^{N}\binom{N}{k}\left(\frac{z}{2}\right)^N p_k + \frac{z}{2}\\
p(z) &= \frac{z}{2} + p(\frac{z}{2})e^{\frac{z}{2}}\\
p(z) &= \frac{z}{2} + \frac{z}{4}e^{\frac{z}{2}} + p(\frac{z}{4})e^{\frac{3z}{4}}\\
p(z) &= \frac{z}{2} + \frac{z}{4}e^{\frac{z}{2}} + \frac{z}{8}e^{\frac{3z}{4}} + p(\frac{z}{8})e^{\frac{7z}{8}}\\
p(z) &= z\sum_{k\geqslant 0}\frac{1}{2^{k+1}}e^{z(1-\frac{1}{2^k})}.
\end{aligned}$$

The Nth coefficient of $p(z)$ is thus

$$p_N = N![z^N]p(z) = N\sum_{k\geqslant 0}\frac{1}{2^{k+1}}\left(1-\frac{1}{2^k}\right)^{N-1}.$$

Note that

$$\begin{aligned}
\left(1-\frac{1}{2^k}\right)^{N-1} &= \exp\left(\log\left(\left(1-\frac{1}{2^k}\right)^{N-1}\right)\right)\\
&= \exp\left((N-1)\left(\frac{-1}{2^k} + O(\frac{1}{2^{2k}})\right)\right)\\
&\sim e^{\frac{-N}{2^k}}.
\end{aligned}$$

And therefore

$$
\begin{aligned}
p_N &= N \sum_{k \geqslant 0} \frac{1}{2^{k+1}} \left(1 - \frac{1}{2^k}\right)^{N-1} \\
&\sim N \sum_{k \geqslant 0} \frac{e^{\frac{-N}{2^k}}}{2^{k+1}}.
\end{aligned}
$$

We will now isolate the periodic terms:

$$
\begin{aligned}
\sum_{k \geqslant 0} \frac{e^{\frac{-N}{2^k}}}{2^{k+1}} &= \sum_{0 \leqslant k < \lfloor \lg N \rfloor} \frac{e^{\frac{-N}{2^k}}}{2^{k+1}} + \sum_{k \geqslant \lfloor \lg N \rfloor} \frac{e^{\frac{-N}{2^k}}}{2^{k+1}} \\
&= \sum_{k < \lfloor \lg N \rfloor} \frac{e^{\frac{-N}{2^k}}}{2^{k+1}} + \sum_{k \geqslant \lfloor \lg N \rfloor} \frac{e^{\frac{-N}{2^k}}}{2^{k+1}} + O\left(e^{-N}\right) \\
&= \sum_{k < 0} \frac{e^{\frac{-N}{2^{k+\lfloor \lg N \rfloor}}}}{2^{k+1+\lfloor \lg N \rfloor}} + \sum_{k \geqslant 0} \frac{e^{\frac{-N}{2^{k+\lfloor \lg N \rfloor}}}}{2^{k+1+\lfloor \lg N \rfloor}} + O\left(e^{-N}\right) \\
&= \sum_{k < 0} e^{-2^{\{\lg N\}-k}} \frac{2^{\{\lg N\}}}{N 2^{k+1}} + \sum_{k \geqslant 0} e^{-2^{\{\lg N\}-k}} \frac{2^{\{\lg N\}}}{N 2^{k+1}} + O\left(e^{-N}\right) \\
&= \frac{1}{N} \left(\sum_{k < 0} e^{-2^{\{\lg N\}-k}} \frac{2^{\{\lg N\}}}{2^{k+1}} + \sum_{k \geqslant 0} e^{-2^{\{\lg N\}-k}} \frac{2^{\{\lg N\}}}{2^{k+1}} \right) + O\left(e^{-N}\right) \\
&= \frac{2^{\{\lg N\}}}{N e^{2^{\{\lg N\}}}} \left(\sum_{k < 0} \frac{1}{e^{2^{-k}} 2^{k+1}} + \sum_{k \geqslant 0} \frac{1}{e^{2^{-k}} 2^{k+1}} \right) + O\left(e^{-N}\right) \\
&\sim \frac{2^{\{\lg N\}}}{N e^{2^{\{\lg N\}}}} \left(\frac{1}{\log(4)} \right) + O\left(e^{-N}\right),
\end{aligned}
$$

where the last equality follows by approximating the sum with an integral (Euler-McLaurin).

So that

$$
p_N = \frac{2^{\{\lg N\}}}{e^{2^{\{\lg N\}}}} \left(\frac{1}{\log(4)} \right) + O\left(e^{-N}\right).
$$