Homework 6: Exercise 9.58

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Let $\mathcal{M}$ be the class of partial mappings and $\mathcal{C}$ the class of rooted Cayley trees.

We can think of a partial mapping as a whole mapping with an additional *null* node, to which all elements with undefined image point. Furthermore, *null* points to itself and is practically a singleton cycle of trees (a.k.a. a single tree). Symbolically, this translates to

$$\mathcal{M} = \text{SET}\left(\text{CYC}\left(\mathcal{C}\right) + \mathcal{C}\right), \quad \mathcal{C} = \mathcal{Z} \times \text{SET}\left(\mathcal{C}\right),$$

where the addition $\mathcal{C}$ corresponds to the tree rooted at the *null* node.

Translating to generating functions, we have

$$M(z) = \exp(\ln(\frac{1}{1 - C(z)} + C(z))$$
$$M(z) = \frac{e^{C(z)}}{1 - C(z)}$$
$$C(z) = z \cdot e^{C(z)}$$

Since there is no explicit form for $C(z)$ we will use the Lagrange-Bürmann inversion theorem to extract coefficients. In this case, $g(z) = C(z)$ and $f(u) = g^{-1}(z) = \frac{u}{e^u}$, and we can check that indeed

$$f(g(z)) = \frac{C(z)}{e^{C(z)}} = z$$
$$f(0) = \frac{0}{e^0} = 0$$
$$f'(0) = \frac{1 - 0}{e^0} = 1 \neq 0.$$

Setting $H(u) = \frac{e^u}{1-u}$, the theorem states

$$[z^n]H(g(z)) = \frac{1}{n}[u^{n-1}]H'(u)\left(\frac{u}{f(u)}\right)^n$$
$$[z^n]\frac{e^{C(z)}}{1 - C(z)} = \frac{1}{n}[u^{n-1}]\frac{e^u(2-u)}{(1-u)^2}\left(\frac{u}{\frac{u}{e^u}}\right)^n$$
$$[z^n]M(z) = \frac{1}{n}[u^{n-1}]\frac{e^u(2-u)}{(1-u)^2}e^{un}$$
$$[z^n]M(z) = \frac{1}{n}[u^{n-1}]\frac{2-u}{(1-u)^2}e^{u(n+1)}.$$

Using the binomial convolution on $\frac{2-u}{(1-u)^2}$ and $e^{u(n+1)}$, we have

$$\begin{aligned}
[z^n]M(z) &= \frac{1}{n}\sum_{k=0}^{n}\frac{\binom{n-1}{k}}{(n-1)!}\left((k-1)![u^{k-1}]\frac{2-u}{(1-u)^2}\right)\left((n-k)![u^{n-k}]\left(e^{u(n+1)}\right)\right) \\
&= \frac{1}{n}\sum_{k=0}^{n}[u^{k-1}]\frac{2-u}{(1-u)^2}[u^{n-k}]\left(e^{u(n+1)}\right)
\end{aligned}$$

Using partial fractions, we can extract coefficients from the left term:

$$\begin{aligned}
[u^{k-1}]\frac{2-u}{(1-u)^2} &= [u^{k-1}]\left(\frac{1}{1-u}+\frac{1}{(1-u)^2}\right) \\
&= [u^{k-1}]\left(\frac{1}{1-u}\right)+[u^{k-1}]\left(\frac{1}{(1-u)^2}\right) \\
&= 1+((k-1)+1) = k+1,
\end{aligned}$$

while the coefficients of the right term follow from Taylor expansion:

$$[u^{n-k}]\left(e^{u(n+1)}\right) = [u^{n-k}]\sum_{i=0}^{\infty}\frac{u^i(n+1)^i}{i!} = \frac{(n+1)^{n-k}}{(n-k)!}.$$

Plugging these back into the binomial convolution, we have

$$\begin{aligned}
[z^n]M(z) &= \frac{1}{n}\sum_{k=0}^{n}[u^{k-1}]\frac{2-u}{(1-u)^2}[u^{n-k}]\left(e^{u(n+1)}\right) \\
&= \frac{1}{n}\sum_{k=0}^{n}(k+1)\frac{(n+1)^{n-k}}{(n-k)!} \\
&\sim \frac{1}{n+1}\sum_{k=0}^{n}(k+1)\frac{(n+1)^{n-k}}{(n-k)!} \\
&= \frac{1}{n+1}\sum_{k=0}^{n}(n+1-(n-k))\frac{(n+1)^{n-k}}{(n-k)!} \\
&= \sum_{k=0}^{n}\frac{(n+1)^{n-k}}{(n-k)!}-\sum_{k=0}^{n-1}\frac{(n+1)^{n-k-1}}{(n-k-1)!} \\
&= \sum_{k=0}^{n}\frac{(n+1)^{n-k}}{(n-k)!}-\sum_{k=1}^{n}\frac{(n+1)^{n-k}}{(n-k)!} \\
&= \frac{(n+1)^n}{n!}.
\end{aligned}$$

Note that to go form the second line to the third line, we used the fact that $n+1$ and n are asymptotically equivalent. It follows that the number of partial mappings of size n is $n!$ times the nth coefficient, *i.e.* $(n+1)^n$.

Combinatorially, we can interpret this number as follows. Each of the n elements in the domain has $n + 1$ options for its image: Either one of the n elements, or undefined. The total number of partial mappings over n elements is then $(n + 1)^n$ by product rule.