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COS 488/MAT 474
Problem Set 6, Q1

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AofA Exercise 8.3 How long should a string of bits be taken to be 50% sure that there are at least 32 consecutive 0s?

Solution. The total number of bitstrings of length N is 2^N, and the number of bitstrings of length N with no run of 32 consecutive 0s is $[z^N]B_{32}(z)$, where

$$B_{32}(z) = \frac{1 - z^{32}}{1 - 2z + z^{33}},$$

the OGF for bitstrings with no run of 32 consecutive 0s. By Theorem 4.1, $[z^N]B_{32}(z) \sim C\beta^N$, where $1/\beta$ is the root of $1 - 2z + z^{33}$ of smallest modulus, and

$$C = \frac{-\beta(1 - (\frac{1}{\beta})^{32})}{-2 + 33(\frac{1}{\beta})^{32}}.$$

The root of $1-2z+z^{33}$ of smallest modulus is very close to $\frac{1}{2}$, so we write $\frac{1}{\beta} = \frac{1}{2}+\epsilon$ and approximate epsilon as follows:

$$\begin{aligned}(\tfrac{1}{2} + \epsilon)^3 3 - 2(\tfrac{1}{2} + \epsilon) + 1 &\approx 0 \\ \Rightarrow (\tfrac{1}{2})^{33} - 2\epsilon &\approx 0 \\ \Rightarrow \epsilon \approx \frac{1}{2^{34}} &\approx 5.82 \times 10^{-11}.\end{aligned}$$

So $\frac{1}{\beta} \approx \frac{1}{2} + \frac{1}{2^{34}} = \frac{2^{33}+1}{2^{34}} \approx \frac{1}{2} + (5.82 \times 10^{-11})$, and thus $\beta \approx \frac{2^{34}}{2^{33}+1} \approx 2 - (2.328 \times 10^{-10})$. We notice that C is very close to 1, so we approximate C by 1.

So far, we've established that

$$\frac{[z^n]B_{32}(z)}{2^N} \approx C\left(\frac{\beta}{2}\right)^N \approx \left(\frac{2^{33}}{2^{33}+1}\right)^N,$$

and we want to solve for N when this quantity is approximately $\frac{1}{2}$:

$$\left(\frac{2^{33}}{2^{33}+1}\right)^N \approx \frac{1}{2}.$$

Taking the log of both sides with base $\frac{2^{33}}{2^{33}+1}$ and then simplifying using log rules yields

$$\begin{aligned}N &\approx \frac{\log_2(\frac{1}{2})}{\log_2(2^{33}) - \log_2(2^{33}+1)} \\ &\approx \frac{-1}{-1.68 \times 10^{-10}} \\ &\approx 5.954 \times 10^9.\end{aligned}$$

Therefore, a bitstring needs to be of length at least $\approx 5.954 \times 10^9$ in order to have a 50% chance of having a run of 32 consecutive 0s.