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COS 488/MAT 474
Problem Set 6, Q3

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AofA Exercise 8.57 Solve the recurrence

$$p_N = \frac{1}{2^N}\sum_{k=0}^{N}\binom{N}{k}p_k \qquad \text{for } N > 1 \text{ with } p_0 = 0 \text{ and } p_1 = 1,$$

to within the oscillating term.

Solution. First, we make the recurrence valid for all N, then multiply by $\frac{z^N}{N!}$ and sum on N to obtain a functional equation for $p(z)$:

$$p_N = \frac{1}{2^N}\sum_{k=0}^{N}\binom{N}{k}p_k + \frac{1}{2}\delta_{N,1}$$

$$\begin{aligned}
p(z) &= \sum_{N=0}^{\infty} p_N \frac{z^N}{N!} \\
&= \sum_{N=0}^{\infty}\left(\sum_{k=0}^{N}\binom{N}{k}\frac{p_k}{2^N}\right)\frac{z^N}{N!} + \frac{z}{2} \\
&= \sum_{N=0}^{\infty}\left(\sum_{k=0}^{N}\binom{N}{k}\frac{p_k}{2^k}\frac{1}{2^{N-k}}\right)\frac{z^N}{N!} + \frac{z}{2} \\
&= p\left(\frac{z}{2}\right)e^{z/2} + \frac{z}{2}.
\end{aligned}$$

We iterate to find an explicit formula for $p(z)$:

$$
\begin{aligned}
p(z) &= \frac{z}{2} + e^{z/2}\left(\frac{z}{4} + e^{z/4} p\left(\frac{z}{4}\right)\right) \\
&= \frac{z}{2} + \frac{z}{4}e^{z/2} + e^{3z/4}\left(\frac{z}{8} + e^{z/8} p\left(\frac{z}{8}\right)\right) \\
&\vdots \\
&= z\sum_{j=0}^{\infty} \frac{1}{2^{j+1}} e^{(1-1/2^j)z} \\
&= z\sum_{j=0}^{\infty} \frac{1}{2^{j+1}} \left(\sum_{N=0}^{\infty} \frac{(1-1/2^j)^N}{N!} z^N\right) && \text{(expand the gf for } e^x\text{)} \\
&= \sum_{N=0}^{\infty} \left(\sum_{j=0}^{\infty} \frac{1}{2^{j+1}} (1-1/2^j)^N\right) \frac{z^{N+1}}{N!} \\
&= \sum_{N=0}^{\infty} \left(\sum_{j=0}^{\infty} \frac{N+1}{2^{j+1}} (1-1/2^j)^N\right) \frac{z^{N+1}}{(N+1)!} \\
&= \sum_{N=1}^{\infty} \left(\sum_{j=0}^{\infty} \frac{N}{2^{j+1}} (1-1/2^j)^N\right) \frac{z^N}{N!}
\end{aligned}
$$

and we extract the coefficients p_N:

$$
\text{for all } N \geq 1, \qquad p_N = \sum_{j=0}^{\infty} \frac{N}{2^{j+1}} \left(1 - \frac{1}{2^j}\right)^{N-1} \sim \boxed{\sum_{j=0}^{\infty} \frac{N}{2^{j+1}} e^{-N/2^j}.}
$$

We can approximate this sum by replacing it with an integral:

$$
\begin{aligned}
p_N \sim \sum_{j=0}^{\infty} \frac{N}{2^{j+1}} e^{-N/2^j} &\sim \int_0^{\infty} \frac{N}{2^{x+1}} e^{-N/2^x}\, dx \\
&= \left[\frac{e^{-N/2^x}}{2\ln 2}\right]_0^{\infty} \\
&= \frac{1-e^{-N}}{2\ln 2} \\
&\to \frac{1}{2\ln 2} && \text{as } N \to \infty.
\end{aligned}
$$

What are the oscillating terms? -1