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COS 488/MAT 474
Problem Set 6, Q4

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AofA Exercise 9.58 Describe the graph structure of *partial mappings*, where the image of a point may be undefined. Set up the corresponding EGF equations and check that the number of partial mappings of size N is $(N+1)^N$.

Solution. The directed graph of a *mapping* is a set of cycles of trees. By contrast, the graph of a *partial mapping* is the union of a set of trees and a set of cycles of trees. The cycles of trees are the graph components in which every atom has an image, and the trees are the graph components in which the root of that tree has no image. With this description, we can describe the set of partial mappings in symbols:

$$\mathcal{PM} = SET(\mathcal{C}) \star SET(CYC(\mathcal{C}))$$

where $\mathcal{C}$ represents the class of Cayley trees (i.e. rooted, unordered, labelled trees) of atoms. As shown on p. 329, the EGF for Cayley trees satisfies the functional equation

$$C(z) = ze^{C(z)}.$$

Therefore, by the symbolic method, the EGF for partial mappings is given by

$$\begin{aligned} PM(z) &= e^{C(z)} \cdot \exp\left(\ln \frac{1}{1-C(z)}\right) \\ &= \frac{C(z)}{z} \cdot \frac{1}{1-C(z)} \\ &= \frac{1}{z} \cdot \frac{C(z)}{1-C(z)}. \end{aligned}$$

Let $g(z) = \frac{z}{1-z} = \sum_{k=1}^{\infty} z^k = \sum_{k=0}^{\infty} g_k z^k$, where $g_0 = 0$ and $g_k = 1$ for all $k \geq 1$. By the Lemma on p. 528, we have

$$\begin{aligned} [z^N]\frac{C(z)}{1-C(z)} &= \sum_{k=0}^{N-1} (N-k)\frac{N^{k-1}}{k!} \\ &= \sum_{k=0}^{N-1} \frac{N^k}{k!} - \sum_{k=1}^{N-1} \frac{N^{k-1}}{(k-1)!} \\ &= \sum_{k=0}^{N-1} \frac{N^k}{k!} - \sum_{k=0}^{N-2} \frac{N^k}{k!} \\ &= \frac{N^{N-1}}{(N-1)!}. \end{aligned}$$

Using this, we can now determine the coefficients of $PM(z)$:

$$[z^N]PM(z) = [z^N]\left(\frac{1}{z} \cdot \frac{C(z)}{1-C(z)}\right) = [z^{N+1}]\frac{C(z)}{1-C(z)} = \frac{(N+1)^N}{N!}.$$

Therefore the number of partial mappings of size N is $(N+1)^N$.