COS 488 - Homework 6 - Question 3

Matt Tyler

5/5

Let $P(z)$ be the EGF corresponding to the sequence p_N, so that $P(z)$ satisfies the equation

$$P(z) = \frac{z}{2} + e^{\frac{z}{2}} P\left(\frac{z}{2}\right).$$

(The $\frac{z}{2}$ term is there to make $P(z)$ agree with $p_0 = 0$ and $p_1 = 1$.) By iteratively applying this recurrence, we have

$$\begin{aligned} P(z) &= \frac{z}{2} + e^{\frac{z}{2}} \left(\frac{z}{4} + e^{\frac{z}{4}} P\left(\frac{z}{4}\right)\right) \\ &= \frac{z}{2} + \frac{z}{4} e^{\frac{z}{2}} + e^{\frac{3z}{4}} \left(\frac{z}{8} + e^{\frac{z}{8}} P\left(\frac{z}{8}\right)\right) \\ &= \sum_{k=0}^{\infty} \frac{e^{z(1-2^{-k})}}{2^{k+1}}. \end{aligned}$$

Therefore, we can extract coefficients to find that

$$\begin{aligned} p_N &= N![z^N]P(z) \\ &= \sum_{k=0}^{\infty} \frac{N}{2^{k+1}} \left(1 - \frac{1}{2^k}\right)^{N-1} \\ &\sim \sum_{k=0}^{\infty} \frac{N}{2^{k+1}} e^{-N/2^k} \\ &= \sum_{k=-\lfloor \lg N \rfloor}^{\infty} \frac{N}{2^{k+1+\lfloor \lg N \rfloor}} e^{-N/2^{k+\lfloor \lg N \rfloor}} \\ &= \sum_{k=-\lfloor \lg N \rfloor}^{\infty} \frac{2^{\{\lg N\}}}{2^{k+1}} e^{-2^{\{\lg N\}-k}}. \end{aligned}$$