COS 488 - Homework 6 - Question 4

5/5

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The graph structure of a partial mapping of N elements is a digraph with N vertices in which each vertex has outdegree at most 1. In other words, the graph structure of a partial mapping consists of connected components in an ordinary matching as well as Cayley trees in which the root vertex does not map to anything.
Let Ξ be the combinatorial class of partial mappings, and let C be the combinatorial class of Cayley. Then, by the above discussion, we have the construction

$$\Xi = SET(C + CYC(C)),$$

which gives the EGF equation

$$\Xi(z) = e^{C(z)+\ln\frac{1}{1-C(z)}} = \frac{e^{C(z)}}{1-C(z)}.$$

Using Lagrange-Burmann inversion with $f(u) = \frac{u}{e^u}$ and $H(u) = \frac{e^u}{1-u}$, we have that

$$\begin{aligned}
[z^N]\Xi(z) &= \frac{1}{N}[u^{N-1}]\frac{e^{(N+1)u}(2-u)}{(1-u)^2} \\
&= \frac{1}{N}\left(2[u^{N-1}]\frac{e^{(N+1)u}}{(1-u)^2} - [u^{N-2}]\frac{e^{(N+1)u}}{(1-u)^2}\right) \\
&= \frac{1}{N}\left(2\sum_{k=0}^{N-1}\frac{(N-K)(N+1)^k}{k!} - \sum_{k=0}^{N-2}\frac{(N-K-1)(N+1)^k}{k!}\right) \\
&= \frac{1}{N}\left(\frac{2(N+1)^{N-1}}{(N-1)!} + \sum_{k=0}^{N-2}\frac{(N+1)^k}{k!}(2(N-k)-(N-k-1))\right) \\
&= \frac{1}{N}\left(\frac{2(N+1)^{N-1}}{(N-1)!} + \sum_{k=0}^{N-2}\frac{(N+1)^k}{k!}((N+1)-k)\right) \\
&= \frac{1}{N}\left(\frac{2(N+1)^{N-1}}{(N-1)!} + \sum_{k=0}^{N-2}\frac{(N+1)^{k+1}}{k!} - \sum_{k=1}^{N-2}\frac{(N+1)^k}{(k-1)!}\right) \\
&= \frac{1}{N}\left(\frac{2(N+1)^{N-1}}{(N-1)!} + \frac{(N+1)^{N-1}}{(N-2)!}\right) \\
&= \frac{(N+1)^N}{N!}.
\end{aligned}$$

Thus, the number of partial mappings of N elements is

$$N![z^n]\Xi(z) = (N+1)^N,$$

as desired.