COS 488 Problem Set #6 Question #3

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Applying the convolution to $p_N = \frac{1}{2^N}\sum_k \binom{N}{k} p_k \implies \frac{p_N}{N!} = \sum_k \frac{1}{2^{N-k}(N-k)!}\frac{p_k}{2^k k!}$ gives that $P(z) = e^{z/2}P(z/2) + \alpha z + \beta$ for some α, β, since the recurrence only applies for $N > 1$. Plugging in $z = 0$ gives $\beta = 0$, so $P(z) = e^{z/2}P(z/2) + \alpha$. Taking the first derivative, we obtain $P'(z) = \frac{1}{2}e^{z/2}(P'(z/2) + P(z/2)) + \alpha$. Plugging in $z = 0$, and noting by assumption $P(0) = p_0 = 0$ and $P'(0) = p_1 = 1$, we have $1 = 1/2(1+0) + \alpha \implies \alpha = 1/2$. As a result, $P(z) = z/2 + e^{z/2}P(z/2)$. Expanding this recurrence, we have

$$P(z) = z/2 + e^{z/2}z/4 + e^{3z/4}z/8 + \ldots = \sum_{j=0}^{\infty} z e^{(1-2^{-j})z} 2^{-j-1}$$

In particular, this yields

$$\frac{p_N}{N!} = \sum_{j=0}^{\infty} \frac{(1-2^{-j})^{N-1}}{(N-1)!} 2^{-j-1}$$

$$p_N = N\sum_{j=0}^{\infty} (1-2^{-j})^{N-1} 2^{-j-1}$$

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I have no clue what to do from here.

You can use a Taylor series approximation which will make the oscillating terms more clear