COS 488 Problem Set #6 Question #4

Tim Ratigan

March 16, 2017

5/5

Considering mappings as the set of cycles of cayley trees, partial mappings can be seen as the set of trees along with the set of cycles of trees, where the latter is the submapping within the partial mapping and the former is trees whose root has no image. This gives the combinatorial construction (if $\mathcal{P}$ corresponds to partial mappings)

$$\mathcal{P} = SET(\mathcal{C}) \star SET(CYC(\mathcal{C}))$$

From the book we know $\mathcal{C} = \mathcal{Z} \star SET(\mathcal{C}) \implies C(z) = ze^{C(z)}$, so

$$\begin{aligned} P(z) &= e^{C(z)} e^{\log \frac{1}{1-C(z)}} \\ &= \frac{C(z)}{z} \frac{1}{1-C(z)} \end{aligned}$$

By the lemma on page 528 of the book, if $g(z) = \frac{z}{1-z}$ so that $g_N = 1$ for $N > 0$, then we have

$$\begin{aligned} [z^n] \frac{C(z)}{1-C(z)} &= \sum_{k=0}^{N-1} (N-k) g_{N-k} \frac{N^{k-1}}{k!} \\ &= \sum_{k=0}^{N-1} (N-k) \frac{N^{k-1}}{k!} \\ &= \sum_{k=0}^{N-1} \frac{N^k}{k!} - \sum_{k=1}^{N-1} \frac{N^{k-1}}{(k-1)!} \\ &= \sum_{k=0}^{N-1} \frac{N^k}{k!} - \sum_{k=0}^{N-2} \frac{N^k}{k!} \\ &= \frac{N^{N-1}}{(N-1)!} \end{aligned}$$

As a result, $P_N = \frac{(N+1)^N}{N!}$, which confirms the number of partial mappings of length N is $(N+1)^N$.