Note II.11: Balls switching chambers: the Ehrenfest model

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Since all balls $b_1,\ b_2,\ ...\ b_N$ begin in A, then we know that after n evolutions, any ball that has moved an odd number of times is in B and that any ball that has moved an even number of times is still in or back in A.

Let us imagine a string such that each character in the string at index i denotes which ball has moved at instant i. So, in such a string, we can see that if a character appears zero or an even number of times, then its corresponding ball is in A, and if it appears an odd number of times, then its corresponding ball is in B. This allows us to construct all words ("balls in A") from an N-letter alphabet where all letters appear an even number of times as

$$E^{[N]}(z) = SEQ_N(\sum_{k>0} SET_{even\ k}(z))$$

$$= (\sum_{k>0}^{N} \tfrac{z^k}{k!})^N where\ k\ is\ even = \tfrac{1}{2}(\sum_{k>0} \tfrac{z^k}{k!} + \sum_{k>0} \tfrac{(-z)^k}{k!})^N$$

$$= (\tfrac{1}{2}(e^z + e^{-z}))^N = (cosh(z))^N$$

We can do something similar for balls ending up in B, which are modeled as words where letters appears an odd number of times.

$$E^{[N]}(z) = SEQ_N(\sum_{k>0} SET_{odd\ k}(z))$$

$$= (\sum_{k>0}^{N} \tfrac{z^k}{k!})^N where\ k\ is\ odd = \tfrac{1}{2}(\sum_{k>0} \tfrac{z^k}{k!} - \sum_{k>0} \tfrac{(-z)^k}{k!})^N$$

$$= (\tfrac{1}{2}(e^z - e^{-z}))^N = (sinh(z))^N$$

We can get the first equation we want to solve for with these two expressions. Assume l balls end up in A, and $N-l$ balls end up in B, or some l letters in the word appear an even number of times while the rest show up an odd number of times. This looks like the binomial distribution thing:

$$E^{[l]}(z) = \binom{N}{l}(cosh(z))^l(sinh(z))^{N-l}$$

At time $2n$, the total number of possible evolutions is N^{2n} and the probability that all the balls are in A is

$$\frac{(2n)!}{N^{2n}}[z^{2n}]E^{[N]}(z) = \frac{(2n)!}{N^{2n}}[z^{2n}](cosh(z))^{N}$$

$$= \frac{(2n)!}{N^{2n}2^{N}}[z^{2n}](e^{z} + e^{-z})^{N}$$

$$= \frac{(2n)!}{N^{2n}2^{N}}[z^{2n}]\sum_{k=0}^{N}\binom{N}{k}e^{z(N-2k)}$$

$$= \frac{(2n)!}{N^{2n}2^{N}}[z^{2n}]\sum_{k=0}^{N}\binom{N}{k}\sum_{m=0}^{N}\frac{(N-2k)^{m}z^{m}}{m!}$$

$$= \frac{(2n)!}{N^{2n}2^{N}}[z^{2n}]\sum_{m=0}^{N}\sum_{k=0}^{N}\binom{N}{k}\frac{(N-2k)^{m}z^{m}}{m!}$$

$$= \frac{(2n)!}{N^{2n}2^{N}}\sum_{k=0}^{N}\binom{N}{k}\frac{(N-2k)^{2n}}{(2n)!}$$

$$= \frac{1}{N^{2n}2^{N}}\sum_{k=0}^{N}\binom{N}{k}(N-2k)^{2n}$$

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