Exam Question 1

Write two OGFs to use in finding the following, for some $M < N$:

a) The number of ways to express N as a sum of at least M positive integers.

b) The number of ways to express N as a sum of at most M positive integers.

Solution

First, define the class of all positive integers of size N and its generating function as

$$I = SEQ_{>0}(z)$$
$$I(z) = \frac{z}{1-z}$$

Let's start with part b. We define the class of all compositions with at most M parts, or a sequence of positive integers that is less than M in length:

$$S = SEQ_{<M}(I)$$

This yields the OGF equation

$$S(z) = I(z) + I(z)^2 + ... + I(z)^M$$

$$S(z) = \sum_{k\geq 1}^{M} (\tfrac{z}{1-z})^k$$

$$S(z) = \frac{z(\frac{z}{1-z})^M - z}{2z-1}$$

For part a, we have a similar definition for the class of compositions with at least M parts:

$$S = SEQ_{>M}(I)$$

$$S(z) = I(z)^M + I(z)^{M+1} + ... + I(z)^N$$

$$S(z) = \sum_{k>M}^{N} (\tfrac{z}{1-z})^k$$

We can use our OGF from part b:

$$S(z) = \sum_{k\geq 1}^{N} (\tfrac{z}{1-z})^k - \sum_{k\geq 1}^{M} (\tfrac{z}{1-z})^k$$

$$S(z) = \frac{1-z}{1-2z} - \frac{z - z(\frac{z}{1-z})^M}{1-2z}$$

$$S(z) = \frac{1-2z-z(\frac{1}{1-z})^M}{1-2z}$$