Analytic Combinatorics Note I.23

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One can design codes with better rates. For example, one can apply the following substitution:

$$\begin{array}{llll} 000 \mapsto 00000 & 001 \mapsto 00001 & 010 \mapsto 00010 & 011 \mapsto 00100 \\ 100 \mapsto 00101 & 101 \mapsto 01000 & 110 \mapsto 01001 & 111 \mapsto 01010 \end{array}$$

Since every 5-bit encoded block begins with 0, it is impossible for two 1-bits to be adjacent. The rate of this encoding is $\frac{5}{3} < 2$.

An allowed code word is the empty word, 1, or either 0 followed by an allowed code word or 10 followed by an allowed code word. If $C(z)$ is the generating function for the allowed code words, we therefore have $C(z) = 1 + z + (z + z^2)C(z)$, so $C(z) = \frac{1+z}{1-z-z^2}$. As discussed earlier (see e.g. Analytic Combinatorics Lecture 8 Slide 7), this means that $C_j = \frac{\phi^2}{\sqrt{5}}\phi^j$. We thus have

$$\begin{aligned} 2^n &\leq \sum_{j=0}^{L} C_j = \sum_{j=0}^{L} \frac{\phi^2}{\sqrt{5}}\phi^j = \frac{\phi^2}{\sqrt{5}} \cdot \frac{\phi^{L+1} - 1}{\phi - 1} \\ n &\leq \lg(\phi^{L+1} - 1) + \lg \frac{\phi^2}{\sqrt{5}(\phi - 1)} = (L+1)\lg\phi + O(1) \\ L + 1 &\geq \frac{n}{\lg\phi} + O(1) \\ L &\geq \frac{n}{\lg\phi} + O(1), \end{aligned}$$

as desired.