Analytic Combinatorics Note II.11

Eric Neyman
4/4/2017 5/5

Let $E_{N,n,\ell}$ be the class of evolutions of N balls $b_1, \ldots, b_n$ ending with ℓ balls in A, with n total moves. Let $W_{N,n,\ell}$ be the class of words with N labeled sets $S_1, \ldots, S_N$ having n objects $j_1, \ldots, j_n$ total, with ℓ of the sets having an even number of objects. Consider the map $\phi : E_{N,n,\ell} \to W_{N,n,\ell}$ sending an evolution that moves ball b_{i_t} $(1 \leq i_t \leq N)$ at time $t - \frac{1}{2}$ $(1 \leq t \leq n)$ to the word where object j_t belongs to the set S_{i_t}. (That is, S_k contains all times when ball b_k is moved (minus $\frac{1}{2}$).) This map's image is indeed inside $W_{N,n,\ell}$, since a ball ends up back in A if and only if it is moved an even number of times, i.e. the corresponding urn has an even number of objects. Consider the map $\phi^{-1} : W_{N,n,\ell} \to E_{N,n,\ell}$ sending a word to an evolution such that if j_t belongs to S_k, then ball b_k moves at time $t - \frac{1}{2}$. This map's image is inside $E_{N,n,\ell}$, since the bins with an even number of objects correspond to the balls that end up back in A. It is clear that ϕ and ϕ^{-1} are in fact inverse maps, which proves that $E_{N,n,\ell}$ is in bijection with $W_{N,n,\ell}$.

Consider the class of N-words (i.e. words with N sets) where the first ℓ sets contain an even number of elements and the last $N - \ell$ sets contain an odd number of elements. This class can be described as

$$SEQ_{\ell}(SET_{\text{even}}(Z)) \star SEQ_{N-\ell}(SET_{\text{odd}}(Z)),$$

which has generating function

$$\left(\frac{e^z + e^{-z}}{2}\right)^{\ell} \times \left(\frac{e^z - e^{-z}}{2}\right)^{\ell}.$$

But *any* of the N sets can be even, not just the first ℓ, so we must multiply by $\binom{N}{\ell}$, giving us the generating function

$$\binom{N}{\ell}(\cosh z)^{\ell}(\sinh z)^{N-\ell},$$

which by our bijection above is equal to $E^{|\ell|}(z)$. In particular, plugging in $\ell = N$ gives

$$E^{|N|}(z) = (\cosh z)^N = 2^{-N}(e^z + e^{-z})^N = 2^{-N}\sum_{k=0}^{N}\binom{N}{k}e^{z(N-k)}e^{-zk} = 2^{-N}\sum_{k=0} N\binom{N}{k}e^{z(N-2k)}$$

$$= 2^{-N}\sum_{k=0}^{N}\binom{N}{k}\sum_{n\geq 0}\frac{(N-2k)^n}{n!}z^n = \sum_{n\geq 0}2^{-N}\sum_{k=0}^{N}\binom{N}{k}\frac{(N-2k)^n}{n!}z^n.$$

Thus, the $2n$-th coefficient of this generating function is

$$2^{-N}\sum_{k=0}^{N}\binom{N}{k}\frac{(N-2k)^{2n}}{(2n)!}.$$

We multiply by $(2n)!$ since this is an EGF; this gives us the total number of ways that urn A ca be full at time $2n$. Dividing by the total number of evolutions of length $2n$, i.e. N^{2n}, gives the probability that urn A is full at time $2n$, i.e.

$$\frac{1}{2^N N^{2n}} \sum_{k=0}^{N} \binom{N}{k} (N-2k)^{2n},$$

as desired.