COS 488 Week 7: Q3

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Dylan Mavrides

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We note that an evolution can be thought of as having N urns, representing the original balls, and n labelled balls, representing moving a ball to the other urn. Thus by putting the n balls in the N urns, we represent a possible evolution. If an urn has an even number of balls in it, then the ball clearly ends in urn A (since they all begin in urn A), thus, we are counting the number of urns containing an even number of balls, across the enumeration.
Another way to think about this, is that we want to have l urns containing an even number of balls, and N-l containing an odd number. Note that there are $\binom{N}{l}$ ways to choose the urns such that this is true.
In general, the EGF for the even values of a generating function is $\frac{f(z)+f(-z)}{2}$ and the EGF for the odd values of a generating function is $\frac{f(z)-f(-z)}{2}$, thus, since we want l urns with even values, and $N-l$ urns with odd values, we have a product of all of these in our generating function, with a coefficient of $\binom{N}{l}$ to account for the l ways we could choose the urns with even values. Thus we have the following generating function:

$$E^{[l]}(z) = \binom{N}{l}(\frac{e^z + e^{-z}}{2})^l(\frac{e^z - e^{-z}}{2})^{N-l} = \binom{N}{l}(\cosh z)^l(\sinh z)^{N-l}$$

as desired.
If we plug in $N = l$ we obtain the other desired equation:

$$E^{[N]}(z) = \binom{N}{N}(\frac{e^z + e^{-z}}{2})^N(\frac{e^z - e^{-z}}{2})^{N-N} = (\cosh z)^N \equiv 2^{-N}(e^z + e^{-z})^N$$

Now, to find the probability that urn A is again full at time 2n, we find the coefficient $E^{[N]}[z^{2n}]$, since $E^{[N]}(z)$ represents the mappings where N balls (all of them) end up in A.

$$2^{-N}(e^z + e^{-z})^N = 2^{-N}\sum_{k=0}^{N}\binom{N}{k}e^{-z(N-2k)}$$

so the coefficient for z^{2n} is

$$2^{-N}\sum_{k=0}^{N}\binom{N}{k}(N-2k)^{2n}$$

(Taylor expand each term in the sum - there would be an $N!$ in the denominator, but since it's an EGF this cancels out.)
There are clearly N^{2n} possible mappings for time $2n$ (because of the bijection I mentioned earlier - we're putting 2n balls into N different possible urns, thus N^{2n}), thus when we divide, we get the desired result for the probability:

$$\frac{1}{2^N N^{2n}} \sum_{k=0}^{N} \binom{N}{k} (N-2k)^{2n}$$

(With David L, Matt T.)