COS 488 Week 6: Q1

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April 5, 2017

Question:
Give the generating function for the number of binary strings that contain neither the bitstring 110 nor 100.

Solution:
We approach the problem in a way similar to how autocorrelation polynomials are used.
We first notationally let $B_{110\&100} = B$. now:

$$B \times \{z_0 + z_1\} + E = B + \{T\}$$

where $\{T\}$ are the set of binary sequences ending in a pattern that includes either of the above patterns (but not the other), or both of them overlapping, except it is not possible for the patterns to both be added by adding a 0 to the end, thus we ignore this case. Thus we see that

$$\{T\} = T_{100} + T_{110}.$$

Now note that we also can obtain the following equations by looking at the internal overlaps of the Ts, and their autocorrelation polynomials. Thus we have:

$$B \times \{z_1 \times z_1 \times z_0\} = T_{110}$$

$$B \times \{z_1 \times z_0 \times z_0\} = T_{100} + T_{110} \times z_0$$

The important thing to note is that when we add the pattern T_{100} to B, we may end up with a term in T_{110}. In fact, for problems slightly more difficult than this (I first tried to make the problem with 1101 and 1011, that was a mess...) we can end up with very complicated expressions here because of many possible overlapping Ts and possible overlapping autocorrelations, etc. It is difficult to count them all correctly.
Now converting to their generating functions, and solving for B, we get:

$$Bz^3 = T_{110}$$

$$Bz^3 = T_{100} + zT_{110}$$

$$2zB + 1 = B + \{T\}$$

And solving for B gives the explicit generating function:

$$B(z) = \frac{1}{-z^4 + 2z^3 - 2z^2 + 1}$$

Which, if you expand, gives the terms 1,2,4,6,9,12... which I checked by hand and were confirmed correct.
I'm unsure if there's a simpler way to do this based on what we've learned so far, but at the very least this seems a reasonable question. If too easy, one can just make the student expand it/possibly asymptotically and that makes it a bit longer. The general idea listed above is also sufficient to solve similar questions as they get more complicated.
Also, my apologies for the slightly late submission. Professor Sedgewick's email listed "Thursday" immediately below this question, and I misunderstood it to mean that it was due Thursday, thus I started this question pretty late.