Homework 7: Note I.23

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The class of valid codes $\mathcal{C}$ is the set of binary strings that do not have any occurrences of 11. Define $\mathcal{P}$ to be the class of binary strings terminating in 11, with no other occurrences of 11 elsewhere. We then have the following symbolic construction

$$\mathcal{C} \times \{\mathcal{Z}_0 + \mathcal{Z}_1\} + E = \mathcal{C} + \mathcal{P}$$
$$\mathcal{C} \times \mathcal{Z}_1 \times \mathcal{Z}_1 = \mathcal{P} + \mathcal{P} \times \mathcal{Z}_1,$$

which translates to the following system of OGFs

$$2zC(z) + 1 = C(z) + P(z)$$
$$z^2C(z) = P(z) + zP(z)$$

Solving the system yields

$$C(z) = \frac{1+z}{1-z-z^2}.$$

Next, we will need to extract coefficients from $C(z)$, which was done in slide 6 of Lecture 8 of Analysis of Algorithms:

$$[z^N]C(z) = \frac{\phi^2}{\sqrt{5}}\phi^N.$$

The number of available code words of length at most L is the sum of coefficients of $z^0, \cdots, z^L$ in $C(z)$.

$$\sum_{i=0}^{L}[z^i]C(z) = \frac{\phi^2}{\sqrt{5}}\phi^i = \frac{\phi^2}{\sqrt{5}}\frac{\phi^{L+1}-1}{\phi-1}.$$

For there to be enough available code words, we need

$$\frac{\phi^2}{\sqrt{5}}\frac{\phi^{L+1}-1}{\phi-1} \geqslant 2^n.$$

Solving for L,

$$(L+1)\lg(\phi) \geqslant n + O(1)$$
$$L \geqslant \frac{n}{\lg(\phi)} + O(1)$$

as desired.