Homework 7: Note II-11

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We can think of each evolution of the system as a balls and bins setup, where each particle corresponds to a bin, and each turn corresponds to a labelled ball that is thrown into a bin. For example, if at turn i particle b is moved from chamber A to chamber B, bin b will include the number i. Note that the particles whose bins have an even number of balls in them end up in chamber A.

The set of possible outcomes of this experiment after n turns is thus given by the number of ways of assigning n balls to N bins. Since we are interested in the parity of the sizes of the bins, we will split up the bins into those with an even number of balls and those with an odd number of balls.

Since there is exactly one way of putting k balls into a bin for all k, the EGF for all bins of even size is

$$\frac{e^z + e^{-z}}{2} = \sum_k \frac{z^{2k}}{(2k)!},$$

corresponding to the counting sequence $1, 0, 1, 0, \cdots$.

Similarly, the EGF for all bins of odd size is

$$\frac{e^z - e^{-z}}{2} = \sum_k \frac{z^{2k-1}}{(2k-1)!},$$

corresponding to the counting sequence $0, 1, 0, 1, \cdots$.

The number of ways of assigning n balls to N bins is the binomial convolution of the above two EGFs, since every such assignments can be uniquely divided into picking some number l of bins to be of even size, populating those l bins with an even number of balls each, and finally populating the remaining $N - n$ bins with an odd number of balls each. Therefore,

$$E(z) = \sum_{l=0}^{N} \binom{N}{l} \left(\frac{e^z + e^{-z}}{2}\right)^l \left(\frac{e^z - e^{-z}}{2}\right)^{N-l} = \sum_{l=0}^{N} \binom{N}{l} (\cosh z)^l (\sinh z)^{N-l}$$

More specifically (and as required by the question), the number of evolutions that result in exactly l evens-sized bins (*i.e.* l balls in chamber A) is given by

$$E^{[l]}(z) = \binom{N}{l} (\cosh z)^l (\sinh z)^{N-l}.$$

Plugging in $l = N$, it immediately follows that

$$E^{[N]}(z) = (\cosh z)^N = 2^{-N}(e^z + e^{-z})^N.$$

Finally, we will compute the probability that chamber A is full at time $2n$. To do so, we note that the total number of possible evolutions after $2n$ steps is N^{2n} (in each step, either of the N balls can be chosen to change chambers). It suffices to divide the number of evolutions of $2n$ turns that result in chamber A being full by the total number of evolutions.

$$\begin{aligned}
[z^{2n}]E^{[N]}(z) &= [z^{2n}]\frac{(e^z + e^{-z})^N}{2^N} \\
&= \frac{1}{2^N}[z^{2n}]\sum_{k=0}^{N}\binom{N}{k}e^{kz}e^{-(N-k)z} \\
&= \frac{1}{2^N}[z^{2n}]\sum_{k=0}^{N}\binom{N}{k}e^{-(N-2k)z} \\
&= \frac{1}{2^N}\sum_{k=0}^{N}\binom{N}{k}(N-2k)^{2n},
\end{aligned}$$

where we have used the binomial expansion in the first line and the Taylor expansion of e^{ax} in the third line.

The probability that all balls end up in chamber A after $2n$ turns is thus

$$\frac{1}{2^N N^{2n}}\sum_{k=0}^{N}\binom{N}{k}(N-2k)^{2n}$$

as desired.