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COS 488/MAT 474
Problem Set 7, Q3

5/5

AC Note II.11 *Balls switching chambers: the Ehrenfest model.* Consider a system of two chambers A and B. There are N distinguishable balls, and, initially, chamber A contains them all. At each instant $t = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$, exactly one ball changes from one chamber to the other. Let $E_n^{[\ell]}$ be the number of possible evolutions that lead to chamber A containing exactly ℓ balls at time $t = n$, and $E^{[\ell]}(z)$ the corresponding EGF. Show that

$$E^{[\ell]}(z) = \binom{N}{\ell}(\cosh z)^{\ell}(\sinh z)^{N-\ell}, \qquad E^{[N]}(z) = (\cosh z)^N \equiv \frac{1}{2^N}(e^z + e^{-z})^N.$$

In particular, show that the probability that chamber A is again full at time $t = 2n$ is

$$\frac{1}{2^N N^{2n}} \sum_{k=0}^{N} \binom{N}{k} (N - 2k)^{2n}.$$

Solution. Let $\mathcal{E}_n^{[\ell]}$ be the class of evolutions of length n that lead to chamber A containing exactly ℓ balls. We can think of such an evolution as putting n tokens (one for each step) into N urns (one for each ball). At each step, we put a token into urn k if ball k is moved from one chamber to the other. Since all the balls begin in chamber A, we can see that a ball ends the evolution in chamber A if and only if it has an even number of tokens in its corresponding urn. So an evolution is in $\mathcal{E}_n^{[\ell]}$ if and only if there are exactly ℓ urns containing an even number of tokens. Such an evolution consists of a length-ℓ sequence of sets of even size, and a length-$(N - \ell)$ sequence of sets of odd size. We can choose any ℓ-element subset to be the sequence of sets of even size. In symbols, we write

$$\mathcal{E}^{[\ell]} = \sum \Big(SEQ_{\ell}(SET_{\text{even}}(Z)) \star SEQ_{N-\ell}(SET_{\text{odd}}(Z)) \Big),$$

where the sum is taken over *all* subsets of N of size ℓ, of which there are $\binom{N}{\ell}$. Translating this into an EGF equation, we get

$$\begin{aligned}
E^{[\ell]}(z) &= \binom{N}{\ell} \left(\left(\sum_{k \text{ even}} \frac{z^k}{k!} \right)^{\ell} \cdot \left(\sum_{k \text{ odd}} \frac{z^k}{k!} \right)^{N-\ell} \right) \\
&= \binom{N}{\ell} \left(\left(\frac{1}{2} \sum_{k=0}^{\infty} \frac{z^k + (-z)^k}{k!} \right)^{\ell} \cdot \left(\frac{1}{2} \sum_{k=0} \frac{z^k - (-z)^k}{k!} \right)^{N-\ell} \right) \\
&= \binom{N}{\ell} \left(\frac{e^z + e^{-z}}{2} \right)^{\ell} \left(\frac{e^z - e^{-z}}{2} \right)^{N-\ell} \\
&= \binom{N}{\ell} (\cosh z)^{\ell} (\sinh z)^{N-\ell}.
\end{aligned}$$

Taking $\ell = N$, the EGF for evolutions where all the balls end in chamber A is

$$\begin{aligned}
E^{[N]}(z) &= (\cosh z)^N \\
&= \frac{1}{2^N}(e^z + e^{-z})^N \\
&= \frac{1}{2^N}\sum_{k=0}^{N}\binom{N}{k} e^{kz}\cdot e^{-(N-k)z} \\
&= \frac{1}{2^N}\sum_{k=0}^{N}\binom{N}{k} e^{(2k-N)z} \\
&= \frac{1}{2^N}\sum_{k=0}^{N}\binom{N}{k}\left(\sum_{n=0}^{\infty}\frac{(2k-N)^n z^n}{n!}\right) \\
&= \sum_{n=0}^{\infty}\left(\frac{1}{2^N}\sum_{k=0}^{N}\binom{N}{k}(2k-N)^n\right)\frac{z^n}{n!} \\
\Rightarrow E_n^{[N]} &= \frac{1}{2^N}\sum_{k=0}^{N}\binom{N}{k}(2k-N)^n.
\end{aligned}$$

The total number of evolutions of length $2n$ is N^{2n}, because for each of the $2n$ steps, there is a choice of which of the N balls to move. Therefore, the probability that chamber A is full at time $2n$ is

$$\frac{E_{2n}^{[N]}}{N^{2n}} = \frac{1}{2^N N^{2n}}\sum_{k=0}^{N}\binom{N}{k}(N-2k)^{2n}.$$