COS 488 - Homework 7 - Question 3

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Let $\mathcal{X}$ and $\mathcal{Y}$ represent the labelled combinatorial classes of urns that contain an even number of objects and an odd number of objects, respectively, so that

$$\mathcal{X}(z) = \frac{e^z + e^{-z}}{2} = \cosh z \text{ and}$$
$$\mathcal{Y}(z) = \frac{e^z + e^{-z}}{2} = \sinh z.$$

Then, there is a natural correspondence between evolutions that lead to chamber A containing l balls at instant n and mappings $\{1, \ldots, n\} \to \{1, \ldots, N\}$ in which the preimage of l points have an even cardinality and the preimage of $N - l$ points have an odd cardinality (given an evolution leading to chamber A containing l balls at instant n, the corresponding mapping sends each $i \in \{1, \ldots, n\}$ to the index of the ball that gets moved in the i^{th} iteration). If we fix the l points whose preimage has an even cardinality, then we can construct the evolutions that lead to chamber A containing l balls at instant n as a sequence of length l of elements of $\mathcal{X}$ followed by a sequence of length $N - l$ of elements of $\mathcal{Y}$. Since there are $\binom{N}{l}$ ways to choose the l points whose preimage has an even cardinality, we have the construction

$$E^{[l]} = \binom{N}{l} SEQ_l(\mathcal{X}) \times SEQ_{N-l}(\mathcal{Y}),$$

which gives the EGF equation

$$E^{[l]}(z) = \binom{N}{l} (\cosh z)^l (\sinh z)^{N-l},$$

as desired.
In particular, when $l = N$, we have

$$E^{[N]}(z) = \binom{N}{N} (\cosh z)^N (\sinh z)^{N-N} = (\cosh z)^N.$$

Therefore, the number of evolutions that lead to chamber A being filled again at time $2n$ is

$$\begin{aligned}
(2n)!\,[z^{2n}]E^{[N]}(z) &= (2n)!\,[z^{2n}](\cosh z)^N \\
&= (2n)!\,[z^{2n}] \left(\frac{e^z + e^{-z}}{2} \right)^N \\
&= (2n)!\,[z^{2n}] \frac{1}{2^N} \sum_{k=0}^{N} \binom{N}{k} (e^z)^{N-k} (e^{-z})^k \\
&= \frac{1}{2^N} \sum_{k=0}^{N} \binom{N}{k} (2n)!\,[z^{2n}] e^{(N-2k)z} \\
&= \frac{1}{2^N} \sum_{k=0}^{N} \binom{N}{k} (N - 2k)^{2n}.
\end{aligned}$$

Since there are N^{2n} different evolutions up to time $2n$ (there are N possibilities for the ball that moves at each of $2n$ steps), the probability that urn A is again full at time $2n$ is

$$\frac{1}{2^N N^{2n}} \sum_{k=0}^{N} \binom{N}{k} (N - 2k)^{2n},$$

as desired.