COS 488 - Homework 7 - Question & Answer

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1 Question

Given an integer $k > 1$, define a rooted ordered tree to be k-ary iff every non-external node has exactly k children. How many k-ary trees are there with k^2 external nodes?
Hint: The answers for $k = 2$ and $k = 3$ are 5 and 55, respectively.

2 Answer

Define T_k to be the combinatorial class of k-ary trees, enumerated by external nodes. Then, a k-ary tree consists of a single external node or an internal root node connected to k other k-ary trees. Therefore, we have the construction

$$T_k = Z_{ext} + Z_{int} \times T_k \times \cdots \times T_k$$

where there are k instances of T_k in the product. Therefore, we have the OGF equation

$$T_k(z) = z + T_k(z)^k.$$

Let $f(u) = u - u^k$, so that we have the equation $f(T_k(z)) = z$. Then, since $f(0) = 0$ and $f'(0) = 1 - k0^{k-1} = 1$ (since $k > 1$), we can use Lagrange inversion to get that

$$[z^{k^2}]T_k(z) = \frac{1}{k^2}[u^{k^2-1}]\left(\frac{1}{1-u^{k-1}}\right)^{k^2}.$$

Then, since

$$\left(\frac{1}{1-u^{k-1}}\right)^{k^2} = \sum_{i=0}^{\infty}\binom{i+k^2-1}{k^2-1}(u^{k-1})^i,$$

we have

$$[z^{k^2}]T_k(z) = \frac{1}{k^2}\binom{(k+1)+(k^2-1)}{k^2-1} = \frac{1}{k^2}\binom{k(k+1)}{k+1} = \frac{1}{k+1}\binom{k(k+1)}{k}.$$

Since

$$\frac{1}{2+1}\binom{2(2+1)}{2} = 5 \text{ and } \frac{1}{3+1}\binom{3(3+1)}{3} = 55,$$

this checks out.

Note that this gives an immediate proof of the slightly non-obvious number-theoretic result that $\binom{k(k+1)}{k}$ is a multiple of $k+1$.