# COS 488 Problem Set #7 Question #3

Tim Ratigan

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We first consider two cases: the generating function for the number of ways 1 ball is in $A$ at time $t$ and the number of ways it is not; that is, computing $E_1^{[0]}$ and $E_1^{[1]}$. Since the ball is in $A$ exactly at even times, $E_1^{[1]} = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} = \cosh z$, and conversely it is in $B$ exactly at odd times, so $E_1^{[0]} = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} = \sinh z$. Now, we wish to compute $E_n^{[\ell]}$. Note that this is simply keeping $\ell$ balls in $A$ and moving $n-\ell$ balls to $B$, so if $\mathcal{C}_{i,A}$ (resp. $\mathcal{C}_{i,B}$) denotes the combinatorial class of sending ball $i$ to $A$ (resp. $B$), and $\mathcal{S}_{n,\ell}$ denotes the combinatorial class of keeping $\ell$ balls in $A$ out of $n$ total, then

$$\begin{aligned}
\mathcal{S}_{n,\ell} &= \sum_{\substack{\sigma\in\mathfrak{S}_n \\ j_1\dots j_\ell = A \\ j_{\ell+1}\dots j_n = B}} \prod_i \mathcal{C}_{i,j_{\sigma(i)}} \\
E_n^{[\ell]} &= \binom{n}{\ell}(E_1^{[1]})^\ell (E_1^{[0]})^{n-\ell} \\
&= \binom{n}{\ell}(\cosh z)^\ell (\sinh z)^{n-\ell}
\end{aligned}$$

When $\ell = n$, this simplifies to $(\cosh z)^n = 2^{-n}(e^z + e^{-z})^n$. Note $(e^z+e^{-z})^n = \sum_{k=0}^n \binom{n}{k} e^{(n-2k)z}$, so the number of ways $A$ has $n$ balls at $t = 2n$ is $2^{-n}\sum_{k=0}^n \binom{n}{k}(n-2k)^{2n}$. However, since there are $n$ balls and $2n$ steps the total number of possible movements is $n^{2n}$, so the probability $A$ is full is

$$\frac{1}{2^n n^{2n}} \sum_{k=0}^{n} \binom{n}{k} (n-2k)^{2n}$$