Note III.17
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5/5

First, lets mark the leaves in the construction of Cayley trees:

$$C(z,u) = z(SET(C(z,u)) - SET_0(C(z,u)) + uSET_0(C(z,u))$$
$$C(z,u) = z(e^{C(z,u)} + u - 1)$$

Perform Langrange Inversion where $f(x) = \frac{x}{e^x+u-1}$ and $f'(x) = \frac{e^x - xe^x + u - 1}{(e^x+u-1)^2}$:

$$n![z^n]C(z,u) = n!\tfrac{1}{n}[x^{n-1}](\tfrac{x}{f(x)})^n$$
$$n![z^n]C(z,u) = n!\tfrac{1}{n}[x^{n-1}](e^x + u - 1)^n$$
$$n![z^n]C(z,u) = n!\tfrac{1}{n}[x^{n-1}]\sum_{k=0}^{n}\tbinom{n}{k}e^{kx}(u-1)^{n-k}$$
$$n![z^n]C(z,u) = n!\tfrac{1}{n}[x^{n-1}]\sum_{k=0}^{n}\tbinom{n}{k}(u-1)^{n-k}\sum_{i=1}^{k}\tfrac{1}{i!}x^ik^i$$
$$n![z^n]C(z,u) = n!\tfrac{1}{n}\sum_{k=0}^{n}\tbinom{n}{k}(u-1)^{n-k}\tfrac{1}{(n-1)!}k^{n-1}$$
$$= (n-1)!\sum_{k=0}^{n}\tbinom{n}{k}(u-1)^{n-k}\tfrac{1}{(n-1)!}k^{n-1}$$
$$= \sum_{k=0}^{n}\tbinom{n}{k}(u-1)^{n-k}k^{n-1}$$

your work on the second page is sufficient to solve this problem?

Now to find the number of size n trees with M leaves:

$$[u^M]n![z^n]C(z,u) = [u^M]\sum_{k=0}^{n}\tbinom{n}{k}(u-1)^{n-k}k^{n-1}$$
$$[u^M]n![z^n]C(z,u) = [u^M]\sum_{k=0}^{n}\tbinom{n}{k}k^{n-1}\sum_{i=0}^{k}\tbinom{n-k}{i}u^i(-1)^{n-i-k}$$
$$= \sum_{k=0}^{n}\tbinom{n}{k}\tbinom{n-k}{M}k^{n-1}(-1)^{n-M-k}$$
$$= \sum_{k=0}^{n}\tbinom{n}{M}\tbinom{n-M}{n-M-k}k^{n-1}(-1)^{n-M-k}$$
$$= \tbinom{n}{M}\sum_{k=0}^{n}\tbinom{n-M}{k}k^{n-1}(-1)^{n-M-k}$$
$$= \tbinom{n}{M}(n-M)!\,\{n-1,\ n-M\}\ \text{by the property}\ M!\{n,\ M\} = \sum_{k}^{n}\tbinom{M}{k}k^n(-1)^{n-M-k}$$
$$= \tfrac{n!}{M!}\{n-1,\ n-M\}$$

This is the total count of Cayley trees with n nodes and M leaf nodes. now to find the cumulative cost. We want to look at the number of Cayley trees with n nodes and k branches. The construction is similar to before:

$$C(z,u) = z(SET(C(z,u)) - SET_k(C(z,u)) + uSET_k(C(z,u))$$
$$C(z,u) = z(e^{C(z,u)} + (u-1)\frac{C(z,u)^k}{k!})$$

didn't need to do anything on first page, this is sufficient

Perform Langrange Inversion where $f(x) = \frac{x}{e^x+(u-1)x^k/k!}$ and $f'(x) = \frac{e^x+(u-1)x^k/k!-x(e^x+(u-1)x^{k-1}/(k-1)!)}{(e^x+(u-1)x^k/k!)^2}$:

$$n![z^n]C(z,u) = (n-1)![x^{n-1}](e^x + (u+1)\frac{x^k}{k!})^n$$
$$n![z^n]C(z,u) = (n-1)![x^{n-1}]\sum_{i=0}^{n}\binom{n}{i}e^{x(n-i)}(u+1)^i\frac{x^{ki}}{k!^i}$$
$$n![z^n]C(z,u) = (n-1)![x^{n-1}]\sum_{i=0}^{n}\binom{n}{i}(\frac{u-1}{k!})^i x^{ki}\sum_{j=0}^{i}\frac{1}{j!}x^j(n-i)^j$$
$$= (n-1)![x^{n-1}]\sum_{i=0}^{n}\binom{n}{i}(\frac{u-1}{k!})^i x^{ki}\sum_{j=0}^{i}\frac{1}{j!}x^{j+ki}(n-i)^j$$
$$= (n-1)!\sum_{i=0}^{n}\binom{n}{i}(\frac{u-1}{k!})^i\frac{1}{n-ki-1}(n-i)^{n-ki-1}$$
$$n![z^n]C_u(z,1) = \frac{n}{k!}\frac{(n-1)!}{(n-k-1)!}(n-1)^{n-k-1}$$
$$n![z^n]C_u(z,1) = n\binom{n-1}{k}(n-1)^{n-k-1}$$

Now divide to find the average number of nodes with k branches:

$$\mu(n,k) = \frac{n}{n^{n-1}}\binom{n-1}{k}(n-1)^{n-k-1}$$
$$\sim \frac{n}{n^{n-1}}\frac{(n-1)^k}{k!}(n-1)^{n-k-1}$$
$$\sim n(1-\frac{1}{n})^{n-1}\frac{1}{k!}$$
$$\sim ne^{-1}\frac{1}{k!}$$