3.5/5

Note III.21
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We can construct the class of all integers as a sequence of decimal digits. Mark every digit in the generating function with its own counter as so:

$$S = SEQ(u_1 Z_1 + u_2 Z_2 + ... + u_9 Z_9)$$
$$S(z) = \frac{1}{1-z(u_1+u_2...+u_9)}$$

If we find the coefficient at $[z^{45} u_1{}^1 u_2{}^2 u_3{}^3 ... u_9{}^9]$, we should get the total number of integers that satisfy our conditions. The expression unrolls nicely.

$$[z^{45} u_1{}^1 u_2{}^2 u_3{}^3 u_4{}^4 u_5{}^5 u_6{}^6 u_7{}^7 u_8{}^8 u_9{}^9] S(z) = [z^{45} u_1{}^1 u_2{}^2 u_3{}^3 u_4{}^4 u_5{}^5 u_6{}^6 u_7{}^7 u_8{}^8 u_9{}^9] \sum_{k=0}^{N} z^k (u_1 + u_2 + ... + u_9)^k$$
$$= [u_1{}^1 u_2{}^2 u_3{}^3 u_4{}^4 u_5{}^5 u_6{}^6 u_7{}^7 u_8{}^8 u_9{}^9](u_1 + u_2 + ... + u_9)^{45}$$
$$= [u_1{}^1 u_2{}^2 u_3{}^3 u_4{}^4 u_5{}^5 u_6{}^6 u_7{}^7 u_8{}^8 u_9{}^9] \sum_{k=0}^{N} \binom{45}{k} u_1{}^k (u_2 + u_3 + ... + u_9)^{45-k}$$
$$= [u_2{}^2 u_3{}^3 u_4{}^4 u_5{}^5 u_6{}^6 u_7{}^7 u_8{}^8 u_9{}^9] 45 (u_2 + u_3 + ... + u_9)^{44}$$
$$= [u_2{}^2 u_3{}^3 u_4{}^4 u_5{}^5 u_6{}^6 u_7{}^7 u_8{}^8 u_9{}^9] 45 \sum_{k=0}^{N} \binom{44}{k} u_2{}^k (u_3 + u_4 + ... + u_9)^{44-k}$$
$$= [u_3{}^3 u_4{}^4 u_5{}^5 u_6{}^6 u_7{}^7 u_8{}^8 u_9{}^9] \frac{45(44)(43)}{2} (u_3 + u_4 + ... + u_9)^{42}$$
$$= [u_3{}^3 u_4{}^4 u_5{}^5 u_6{}^6 u_7{}^7 u_8{}^8 u_9{}^9] \frac{45(44)(43)}{2} \sum_{k=0}^{N} \binom{42}{k} u_3{}^k (u_4 + u_5 + ... + u_9)^{42-k}$$
$$= [u_4{}^4 u_5{}^5 u_6{}^6 u_7{}^7 u_8{}^8 u_9{}^9] \frac{45(44)(43)(42)(41)(40)}{2!(3!)} (u_4 + u_5 + ... + u_9)^{39}$$
$$= [u_4{}^4 u_5{}^5 u_6{}^6 u_7{}^7 u_8{}^8 u_9{}^9] \frac{45(44)(43)(42)(41)(40)}{2!(3!)} \sum_{k=0}^{N} \binom{39}{k} u_4{}^k (u_5 + u_6 + ... + u_9)^{39-k}$$
$$= [u_5{}^5 u_6{}^6 u_7{}^7 u_8{}^8 u_9{}^9] \frac{45(44)(43)(42)(41)(40)(39)(38)(37)(36)}{2!(3!)(4!)} (u_5 + ... + u_9)^{35}$$
$$= \frac{45!}{2!3!4!5!6!7!8!9!}$$

We can multiply the enumeration by the number of times each digit appears and its value to get

$$\frac{45!}{2!3!4!5!6!7!8!9!}(1^2 + 2^2 + ... + 9^2)$$
$$= \frac{(285)45!}{2!3!4!5!6!7!8!9!}$$

If we divide this total by the length of the integer, we have the sum of all digits in a single position. Multiply that value by the sum of all powers of ten within 45 digits to get

You want to divide by 45 here, otherwise you're overcounting because each digit takes only gets multiplied by its power of 10, not the sum of every power of ten

$$\frac{(285)44!}{2!3!4!5!6!7!8!9!} \sum_{i=0}^{44} 10^i$$

$$= 4587555960000615321908476928639999999999999999541244403999938467809152307136000000$$

This should be the value from the question.

As to why there are so many nines in there, if a number A has N digits, then the number $(10^M - 1)A$ will have a run of nines at the place values from 10^N to 10^M. This is because $(10^M - 1)A = 10^M A - A$ where the first term $10^M A$ has m - n zeroes at the end, and subtracting A from that term will result in a run of nines. Our giant sum S is such a $(10^M - 1)A$ where $M = 45$ and $A = 45875559600006153219084769286400000$, a 35-digit number.

-0.5pt, how did you get A?