Analytic Combinatorics Note III.17

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We work in full generality, i.e. in terms of nodes of outdegree k. A Cayley tree is a root together with a set of Cayley trees. The set either has cardinality k — in which case the root has outdegree k — or a cardinality other than k — in which case the root does not have outdegree k. The outdegree of the nodes of the sub-trees is not affected by being attached to the root. Thus, if $T(z, u)$ is the generating function for Cayley trees with cost equal to the number of nodes with outdegree k (with k implicit), we have the generating function equation

$$T(z,u) = ze^{T(z,u)} - z\frac{T(z,u)^k}{k!} + uz\frac{T(z,u)^k}{k!} = z\left(e^{T(z,u)} + (u-1)\frac{T(z,u)^k}{k!}\right).$$

This is because $e^{T(z,u)}$ is the expression for a set of trees, but we have to subtract the term for sets of k sub-trees, and at it back in with an extra u (since the root has outdegree k so the exponent of u should increase).

Observe that if $f(y) = \frac{y}{e^y+(u-1)\frac{y^k}{k!}}$, then

$$f(T(z,u)) = \frac{T(z,u)}{e^{T(z,u)} + (u-1)\frac{T(z,u)^k}{k!}} = z.$$

Applying the Lagrange inversion formula, we have

$$[z^n]T(z,u) = \frac{1}{n}[y^{n-1}]\left(\frac{y}{f(y)}\right)^n = \frac{1}{n}[y^{n-1}]\left(e^y + (u-1)\frac{y^k}{k!}\right)^n.$$

Now we differentiate with respect to u. Using $T_u(z,u)$ to denote the partial of $T(z,u)$ with respect to u, we have

$$\begin{aligned}\frac{\partial}{\partial u}[z^n]T(z,u) = [z^n]T_u(z,u) &= \frac{1}{n}[y^{n-1}]\frac{\partial}{\partial u}\left(e^y + (u-1)\frac{y^k}{k!}\right)^n \\ &= \frac{1}{n}[y^{n-1}]n\left(e^y + (u-1)\frac{y^k}{k!}\right)^{n-1}\cdot\frac{y^k}{k!} = [y^{n-1}]\left(e^y + (u-1)\frac{y^k}{k!}\right)^{n-1}\cdot\frac{y^k}{k!}.\end{aligned}$$

Plugging in $u = 1$, we have

$$\frac{\partial}{\partial u}[z^n]T(z,u)\mid_{u=1} = [y^{n-1}]e^{y(n-1)}\cdot\frac{y^k}{k!} = \frac{1}{k!}[y^{n-k-1}]e^{y(n-1)} = \frac{(n-1)^{n-k-1}}{k!(n-k-1)!}.$$

Recall that this is an EGF, so we multiply by $n!$ to get the total number of vertices of outdegree k over all Cayley trees with n nodes. Now, there are n^{n-1} total Cayley trees with n nodes (this was proved in an earlier lecture), so to obtain the average number of vertices with outdegree k in a Cayley tree with n nodes, we divide by n^{n-1}. The average number of vertices with outdegree k is thus

$$\frac{n!(n-1)^{n-k-1}}{k!(n-k-1)!n^{n-1}} = n\binom{n-1}{k}\frac{(n-1)^{n-k-1}}{n^{n-1}} \sim \frac{n(n-1)^k}{k!}\cdot\frac{(n-1)^{n-k-1}}{n^{n-1}} = \frac{n}{k!}\cdot\left(1-\frac{1}{n}\right)^{n-1}$$

$$= \frac{n}{k!} \cdot \left(1 - \frac{1}{n}\right)^n \cdot \frac{n}{n-1} \sim \frac{n}{k!} \cdot \frac{1}{e} = n \cdot e^{-1} \frac{1}{k!},$$

as desired.