Analytic Combinatorics Note III.21

Eric Neyman
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Let P be the set of all numbers with one 1, two 2's, and so on. Note that S is equal to the sum from $k = 0$ to 44 of 10^k times the sum over all of all numbers $n \in P$ of the digit of n in the 10^k-place. That is, we can find S by finding the sum of all the ones-digits, plus ten times the sum of all the tens-digits, and so on.

How many numbers in P have d as their k-th place digit, for some given d and k? There are 44! ways to order the remaining digits, but remembering that copies of the same digit are indistinguishable, we divide by $1!2!\dots(d-1)!(d-1)!(d+1)!\dots 9!$, where we have $(d-1)!$ instead of $d!$ because one d is already placed (in position k). Thus, there are $\frac{d\cdot 44!}{1!2!\dots 9!}$ such numbers. Thus, our way of summing the numbers above gives us

$$S = \sum_{k=0}^{44} 10^k \sum_{d=1}^{9} d\cdot \frac{d \cdot 44!}{1!2!\dots 9!} = \sum_{k=0}^{44} 10^k \cdot \frac{44!}{1!\dots 9!} \sum_{d=1}^{9} d^2 = \frac{285 \cdot 44!}{1!\dots 9!} \sum_{k=0}^{44} 10^k = (10^{45}-1)\cdot \frac{285 \cdot 44!}{9(1!\dots 9!)}.$$

This evaluates to the desired number. The form above gives insight into the nines: what we really have is a large number, i.e. $\frac{285\cdot 44!}{9(1!\dots 9!)}$ (we will call this number N), times 10^{45}, minus N. It turns out that $N \approx 4.6 \times 10^{34}$. Thus, $N \cdot 10^{45}$ has 45 trailing zeros, and when N is subtracted from this number, the trailing zeros not consumed by N (that is, all but the ~ 34 trailing zeros) become nines. But since there are more trailing zeros — forty-five of them — we should be left with a large number of nines in the middle of our number, and indeed, that is exactly what happens.