Analytic Combinatorics Homework 8 Question and Answer

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Define the cost of a ternary (base 3) string to be the sum of its digits.

(a) Write an OBGF for ternary strings with the cost above.

(b) Using the OBGF, compute the average cost of a length-N bit string.

(a) A ternary string is either an empty string or a ternary string plus a 0, 1, or 2. This gives us the OBGF

$$T(z) = 1 + z(1 + u + u^2)T(z).$$

Therefore we have

$$T(z) = \frac{1}{1 - z(1 + u + u^2)}.$$

(b) We have

$$T_u(z, 1) = \frac{z(1 + 2u)}{(1 - z(1 + u + u^2))^2} \Big|_{u=1} = \frac{3z}{(1 - 3z)^2}.$$

The N-th coefficient of this generating function is 3 times the $(N-1)$-th coefficient of the generating function of $\frac{1}{(1-3z)^2} = (1 + 3z + 9z^2 + \dots)^2$. In this convolution we see that the $(N-1)$-th coefficient equals $1 \cdot 3^{N-1} + 3 \cdot 3^{N-2} + \dots + 3^{N-1} \cdot 1 = N \cdot 3^{N-1}$. Therefore we have

$$[z^N]T_u(z, 1) = 3(N \cdot 3^{N-1}) = N \cdot 3^N.$$

We have

$$[z^N]P(z, 1) = [z^N]\frac{1}{1 - 3z} = 3^N,$$

so the mean cost of length-N strings is $\mu_N = \frac{N \cdot 3^N}{3^N} = N.$