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COS 488 Week 7: Q1

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We begin with the symbolic representation. Note that we want Cayley trees, but then we remove ones with indegree k (the question says outdegree, but since outdegree is always 1, I assume this is a typo), then add them back in with the cost variable attached.

$$T = z \times (SET(T) - SET_k(T) + uSET_k(T)$$

$$T(z,u) = z(e^{T(z,u)} - \frac{(T(z,u))^k}{k!} + \frac{u(T(z,u))^k}{k!})$$

We now apply Lagrange inversion with $g = T(z,u)$ and $f(y) = \frac{y}{e^y+(u-1)y^k/k!}$. (Note that we thus have f(g(y)) = z, since we simply solved our above equation for z to obtain this set of f and g.) We now see that:

$$[z^n]T(z,u) = \frac{1}{n}[y^{n-1}](\frac{y}{\frac{y}{e^y+(u-1)y^k/k!}} = \frac{1}{n}[y^{n-1}](e^y + \frac{(u-1)y^k}{k!})^n$$

Now in the following steps, we take the partial derivative with respect to u, plug in 1 for u, then find the coefficient of y^{n-1} and thus z^n, then multiply by $n!$ since it's an EGF, and then finally divide by the total number of Cayley trees n^{n-1} to get the mean number of nodes with indegree k. The equations proceed as follows:

$$[y^{n-1}]T_u(z,u) = [y^{n-1}](e^y + \frac{(u-1)y^k}{k!})^{n-1}\frac{y^k}{k!}$$

$$[y^{n-1}]T_u(z,1) = [y^{n-1}](e^y)^{(}n-1)\frac{y^k}{k!} = [y^{n-1}]e^{y(n-1)}\frac{y^k}{k!}$$

$$= (1 + (y(n-1)) + \frac{(y(n-1))^2}{2!} + ...)\frac{y^k}{k!} = [y^{n-1}]\frac{y^k(y(n-1))^{n-k-1}}{(n-k-1)!k!}$$

$$= \frac{(n-1)^{n-k-1}}{(n-k-1)!k!}$$

And thus to get the mean:

$$\frac{n![z^n]T_u(z,1)}{n^{n-1}[z^n]T(z,1)} = \frac{n!(n-1)^{n-k-1}}{(n-k-1)!k!n^{n-1}} = \binom{n-1}{k}\frac{(n-1)^{n-k-1}}{n^{n-1}}$$

We now apply our asymptotic approximation for combinations (from the AofA website):

$$\sim \frac{(n-1)^k}{k!}\frac{(n-1)^{n-k-1}}{n^{n-2}} = \frac{(n-1)^{n-1}}{n^{n-1}}\frac{n}{k!} = \frac{(1-1/n)^n}{1-1/n}\frac{n}{k!}$$

and in the limit case, the first term goes to 1/e (note that the denominator clearly goes to 1, and the numerator goes to 1/e. This is a well-known identity, but can easily be shown by showing that $(1-1/n)^n(1+1/n)^n \sim 1$ because of how the expansion works out, or you can substitute with -n. Anyway, we then get:

$$\sim \frac{n}{ek!} = n \cdot e^{-1}\frac{1}{k!}$$

as desired.