3.5/5

COS 488 Week 7: Q2

Dylan Mavrides

April 14, 2017

We begin with a symbolic construction representing the situation. We will make a generating function of sequences of digits with markers u_1 through u_9.

$$N = SEQ(u_1 z + u_2 z + ... + u_9 z)$$

$$N(z, u_1, u_2, ..., u_9) = \frac{1}{1 - z(u_1 + ... + u_9)}$$

Thus by pages 186 and 187 in the textbook, since we're looking for the numbers with 45 digits, we have that

$$[u_1^1 u_2^2 ... u_9^9](u_1 + u_2 + ... + u_9)^{45} = \frac{45!}{1!2!...9!}$$

thus we know that there exist this number of numbers of the form described.
Now if you think about adding *all* of the digits, you would want to multiply this number by the sum of the digits in a given number (which is the sum from i = 1 to 9 of i^2). Thus to find the sum of the digits in a single decimal place, we want to divide this sum of squares by 45, then multiply by the number given by the generating function above. Then to get the sum S, we must multiply the sum for a single digit by 11111...111 (with 45 1s). Doing this gives

$$\frac{285}{45} \frac{45!}{1!2!...9!} 111...111$$

which gives the answer provided, when evaluated with Mathematica.
To explain the lot of 9s, we first note that multiplying a number by 111111111... essentially causes a sum of digits. We first note that 285*111111... would give a bunch of 6s (since 2+8+5+1 = 15 and 1+5 = 6 to account for carrying). Now we note that multiplying a bunch of 6s by a number either gives a bunch of 3s, 6s, or 9s (1/3rd of each). A similar situation happens for 3, but if you ever get a bunch of 9s repeating, multiplying it by anything continues to give a bunch of 9s (since 18 gives 1+8, 27 gives 2+7, etc.). Thus all we need is for one non-285, non-1111... terms to have two 3s in their prime factorization (because ...666666666...*3 ...99999... Note that in the more general case, any repeating number * 9 gives repeating 9s). Indeed

-1.5pt, not all multiples of 111...11 have a run of 0s even if they are also multiples of 99..99,, and your intuition does not explain why this one has a lot of this is the case, since we just look at the number of 3s in the prime factorization of the numerator and the denominator. There are 13 3s in the denominator, not counting 45, which we cancel out (1 from 3!, 1 from 4!... 2 from 6! ... 4 from 9!). These can be cancelled out by the digits between 1 and 27 inclusive, in the numerator (yes, I counted the prime factors). Thus there are clearly at least two prime factors of 3, explaining why it works out.

Maybe this doesn't, on the surface, look simple, but with a less rigorous argument - lots of 1s multiplied by some prime factors terminates at repeating 9s - it can still be easily intuitively seen to be true.

Worked with Matt T, Eric N.