COS 488 Week 7: QA1

Dylan Mavrides

April 12, 2017

Question:
In the set of ordered partitions of an integer N, what is the total number of 1s, asymptotically? What is the expected number of 1s, asymptotically, for a randomly selected partition?

Solution:
We first need to redefine our combinatorial class I split into two parts: $I_{\neg 1}$ and I_1. $I_{\neg 1} = SEQ_{>1}(\cdot) \implies I_{\neg 1}(z) = \frac{z^2}{1-z}$, and $I_1 = SEQ_1(\cdot) \implies I_1(z) = z$.
Now we define our BGF in terms of C, the class of all compositions, with size $|c|$, the number of $\cdot$s in c, and parameter $ones(c)$, the number of parts of size 1 in c.
We thus have the construction:

$$C = SEQ(uI_1 + I_{\neg 1})$$

giving the equation

$$C(z,u) = \frac{1}{1-(zu+\frac{z^2}{1-z})} = \frac{1-z}{1-(1+u)z-(1-u)z^2}$$

We now find the enumeration:

$$[z^N]C(z,1) = [z^N]\frac{1-z}{1-2z} = 2^{N-1}$$

and the cumulated cost (by taking the partial with respect to u, then setting u to zero)

$$[z^N]C_u(z,1) = [z^N]\frac{(z-1)^2 z}{(1-2z)^2}$$

We now apply the rational function theorem, with $\nu = 2$ and $1/\beta = 1/2$, which (very nicely) works out to give

$$N2^{N-3}$$

as our asymptotic approximation for the total number of 1s in the partitions of an integer N.
Side note: to verify, I both checked the sequence on OEIS (it exists and matches my interpretation), and then expanded the coefficients out to 100 terms on Mathematica. The explicit 100th term is "16162545152909924869082965868544," and if we divide our approximation at N = 100 by this, we get the ratio .98 (50/51), demonstrating the accuracy of/verifying our approximation.
Thus the expected number of 1s in a random partition of size N is approximately:

$$N2^{N-3}/2^{N-1} = N/4$$

a nice result.

(I couldn't find this result anywhere in the textbook, but it's possible I missed something.)