Homework 8: Note III.17

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We will compute the mean number of nodes of outdegree k in a random Cayley tree. Let z mark the total number of nodes and u mark the nodes with outdegree k. The BGF for Cayley trees is

$$T(z,u) = z\left(e^{T(z,u)} + (u-1)\frac{T(z,u)^k}{k!}\right)$$

Using Lagrange inversion with $f(x,u) = \frac{x}{e^x+(u-1)\frac{x^k}{k!}}$, we have

$$\begin{aligned}
[z^n]T(z,u) &= \frac{1}{n}[x^{n-1}]\left(\frac{x}{f(x)}\right)^n \\
&= \frac{1}{n}[x^{n-1}]\left(e^x + (u-1)\frac{x^k}{k!}\right)^n \\
[z^n]T_u(z,u) &= [x^{n-1}]\left(e^x + (u-1)\frac{x^k}{k!}\right)^{n-1}\frac{x^k}{k!}
\end{aligned}$$

The cumulated cost is given by

$$\begin{aligned}
[z^n]T_u(z,1) &= [x^{n-1}]e^{x(n-1)}\frac{x^k}{k!} \\
&= [x^{n-k-1}]\frac{e^{x(n-1)}}{k!} \\
&= \frac{(n-1)^{n-k-1}}{(n-k-1)!k!},
\end{aligned}$$

while the enumeration is given by

$$[z^n]T(z,1) = \frac{1}{n}[x^{n-1}]e^{xn} = \frac{n^{n-2}}{(n-1)!}.$$

The average number of nodes of degree k is therefore

$$\begin{aligned}
\frac{n![z^n]T_u(z,1)}{n![z^n]T(z,1)} &= \frac{\frac{(n-1)^{n-k-1}}{(n-k-1)!k!}}{\frac{n^{n-2}}{(n-1)!}} \\
&= \binom{n-1}{k}\frac{(n-1)^{n-k-1}}{n^{n-2}}
\end{aligned}$$

Using the asymptotic approximation to the binomial coefficients for $k = O(1)$ as listed on the asymptotics section of the booksite, we have

$$\begin{aligned}
\binom{n-1}{k}\frac{(n-1)^{n-k-1}}{n^{n-2}} &= \frac{(n-1)^k}{k!}\left(1+O(\frac{1}{n})\right)\frac{(n-1)^{n-k-1}}{n^{n-2}} \\
&= \frac{(n-1)^{n-1}}{k!n^{n-2}}\left(1+O(\frac{1}{n})\right) \\
&= \frac{n}{k!}\left(1-\frac{1}{n}\right)^{n-1}\left(1+O(\frac{1}{n})\right) \\
&= n\cdot e^{-1}\frac{1}{k!}
\end{aligned}$$

as desired.