# Homework 8: Note III.21

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An integer (with no zeros) can be thought of as a word from the alphabet $\mathcal{A}$ of the digits 1 through 9, which can symbolically be specified as $\textsc{Seq}(\mathcal{A})$. We will use marker variables $u_1, u_2, \cdots, u_9$ to mark the occurrences of each digit, yielding the following MGF for integers with no zeros

$$\mathcal{I} = \textsc{Seq}\,(\mathcal{A})$$
$$I(z, \mathbf{u}) = \frac{1}{1-(u_1+u_2+\cdots+u_9)z}.$$

The number of integers of the specified form (having exactly one 1, two 2s, *etc.*) can be extracted as the appropriate coefficient of $I(z)$, which is the multinomial coefficient as described on page 187 of the Analytic Combinatorics book):

$$\left[z^{45}u_1^1u_2^2\cdots u_9^9\right] I(z,\mathbf{u}) = \binom{45}{1,2,\cdots,9} = \frac{45!}{1!2!\cdots 9!} := W.$$

This makes sense combinatorially: Integers with one 1, two 2s, three 3s, *etc.* can be constructed by first lining up all the $1+2+\cdots+9 = 45$ digits, then dividing by the number of ways we can permute the same digit.

We have computed the number of such integers above as a multinomial coefficient $W$. We note that the sum of the digits of any such integer is $1+2^2+3^3+\cdots+9^9 = 285$, so the sum of all the digits of all such integers is the product $285W$. Finally, dividing that by the 45 possible positions in every such number, we obtain the sum of the digits in any given position in all such integers (*e.g.* the sum of the rightmost digit in all such integers, which, by symmetry, should be the same for every position): $285W/45 = \frac{19}{3}W$.

The process of summing all these integers can be broken down into summing the digits in every position:

$$S = \frac{19}{3}W(\sum_{i=0}^{44} 10^i) = \frac{19}{3}\frac{45!}{1!2!\cdots 9!}(\sum_{i=0}^{44} 10^i).$$

Note that multiplying a number $x$ by $\sum_{i=0}^{k-1} 10^i$ can be thought of as adding the $x$ to itself $k$ times, shifted left every time. After moving left from the rightmost position for long enough, we get to a point where the given digit represents the iterative sum of the digits of $x$ (the result of summing the digits of $x$ and repeating the process recursively until we are left with a single digit).

-1pt, what about carrying? this kind of works but it's ugly and needs to be explained through

We observe that what is getting multiplied by $\sum_{i=0}^{44} 10^i$ here is divisible by $9$. This is the case since the exponent of 3 in the factorization of the denominator of the multinomial coefficient 13, while the exponent of 3 in the numerator is at least 45/3=15. Therefore, the iterated digit-summing process will result in the singleton digit 9 (since the sum of the digits of a number has the same remainder modulo 9 as does the original number), explaining the occurrence of the long run of 9s in the middle of $S$.