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COS 488 - Homework 8 - Note III.17

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We will prove that the mean number of nodes with indegree k in a random Cayley tree of size n is asymptotic to $\frac{n}{ek!}$. Since a leaf is the same as a node with indegree 0, it will follow immediately that the mean number of leaves in a random Cayley tree of size n is $\frac{n}{e0!} = \frac{n}{e}$.
Let $T_k(z,u)$ denote the bivariate EGF for which $[u^m]n![z^n]T_k(z,u)$ is the number of Cayley trees of size n with m nodes with indegree k. Since a Cayley tree consists of either a root vertex connected to a set of l Cayley trees for some $l \neq k$ or a root vertex connected to a set k Cayley trees, we have the following construction:

$$T_k = Z \times SET_{\neq k}(T_k) + Z \times SET_k(T_k).$$

This gives the bivariate EGF equation

$$T_k(z,u) = z\left(e^{T_k(z,u)} + (u-1)\frac{T_k(z,u)}{k!}\right),$$

so if $f(y) = \frac{y}{e^y + (u-1)y^k/k!}$, then $f(T_k(z,u)) = z$. Since $f(0) = 0$ and $f'(0) = 1 \neq 0$, we can use Lagrange inversion o find that

$$[z^n]T_k(z,u) = \frac{1}{n}[y^{n-1}]\left(\frac{y}{f(y))}\right)^n = \frac{1}{n}[y^{n-1}]\left(e^y + (u-1)\frac{y^k}{k!}\right)^n.$$

Therefore (for all sufficiently large k),

$$\begin{aligned}\frac{\partial}{\partial u}(n![z^n]T_k(z,u))|_{u=1} &= \frac{n!}{n}[y^{n-1}]\left(n\frac{y^k}{k!}\left(e^y + (1-1)\frac{y^k}{k!}\right)^{n-1}\right)\\ &= n![y^{n-1}]\left(\frac{y^k}{k!}e^{y(n-1)}\right)\\ &= n!\left(\frac{1}{k!}\frac{(n-1)^{n-1-k}}{(n-1-k)!}\right)\\ &= n\binom{n-1}{k}(n-1)^{n-1-k}.\end{aligned}$$

Since there are n^{n-1} Cayley trees of size n, the average number of nodes with indegree k in a random Cayley tree is

$$\frac{\frac{\partial}{\partial u}(n![z^n]T_k(z,u))|_{u=1}}{n^{n-1}} = \frac{n}{(n-1)^k}\binom{n-1}{k}\left(1-\frac{1}{n}\right)^{n-1}.$$

Since $\binom{n-1}{k}$ is asymptotic to $\frac{(n-1)^k}{k!}$ and $\left(1-\frac{1}{n}\right)^{n-1}$ is asymptotic to $\frac{1}{e}$, this average is asymptotic to

$$\frac{n}{(n-1)^k}\frac{(n-1)^k}{k!}\frac{1}{e} = \frac{n}{ek!},$$

as desired.