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COS 488 - Homework 8 - Note III.21

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Let $\mathcal{L}$ be the combinatorial class of all strings consisting of the digits between 1 and 9 (inclusive), so that we have the construction

$$\mathcal{L} = SEQ(Z_1 + \cdots + Z_9).$$

This gives the EGF equation

$$\mathcal{L}(z_1, \ldots, z_n) = \frac{1}{1 - (z_1 + \cdots + z_9)}.$$

Therefore, the number of numbers consisting of one instance of the digit 1, two instances of the number 2, and so on is

$$[z_1 z_2^2 \ldots z_9^9]\mathcal{L}(z_1, \ldots, z_n) = [z_1 \ldots z_9^9](z_1 + \cdots + z_9)^{45} = \frac{45!}{1!2! \cdots 9!}.$$

In such a number, the average value of each digit is

$$\frac{1^1 + \cdots + 9^2}{1 + \cdots + 9} = \frac{285}{45} = \frac{19}{3},$$

so the sum S of all such numbers is

$$\frac{10^{45} - 1}{9} \times \frac{19}{3} \times \frac{45!}{1!2! \cdots 9!} = (10^{45} - 1)\frac{19}{27} \times \frac{45!}{1!2! \cdots 9!}.$$

Let

$$T = \frac{19}{27} \times \frac{45!}{1!2! \cdots 9!} = 45875559600006153219084769286400000,$$

so that $S = 10^{45}T - T$. Then, since T terminates in 5 0's, $10^{45}T$ terminates in 50 0's, so since T consists of 35 digits, we should expect S to contain 15 0's at indices (starting from 1 at the right) 36 through 50, and indeed that is the case since

$$S = 4587555960000615321908476928639999999999999999954124440399993846780915230713600000.$$