COS 488 - Homework 8 - Question & Answer

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1 Question

Define the unit N-hypercube to be the set of points $[0,1]^N \subset \mathbb{R}^N$. For example, the unit 0-hypercube is a point, and the unit 3-hypercube is the unit cube. Define a k-face of the unit N-hypercube to be a copy of the k-hypercube in the exterior of the N-hypercube. More formally, a k-face of the unit N-hypercube is a set of the form

$$\prod_{i=1}^{N} S_i$$

where S_i is either $\{0\}$, $\{1\}$, or $[0,1]$ for each $1 \leq i \leq N$, and there are exactly k indices i such that $S_i = [0,1]$.

1. Let $\Xi_{N,k}$ be the number of k-faces of the unit N-hypercube, and let

   $$\Xi(z,u) = \sum_{N=0}^{\infty} \sum_{k=0}^{\infty} \Xi_{N,k} z^N u^k.$$

   Find an explicit form for $\Xi(z,u)$.

2. How many k-faces does the unit N-hypercube have?

3. What is the expected value of the dimension of a random face of the unit N-hypercube (where the dimension of a face can be any value between 0 and N)?

2 Answer

1. We can construct the unit N-hypercube be taking two copies of the unit $(N-1)$-hypercube and joining corresponding points along a unit interval. With this construction, a k-face of the unit N-hypercube is either:

   (a) The 0-face of the unit 0-hypercube,

   (b) A k-face of one of the two copies of the unit $(N-1)$-hypercube, or

   (c) Two corresponding $(k-1)$-faces of the two copies of the unit $(N-1)$-hypercube joined along a unit interval.

   a k-face of one of the two copies of the unit $(N-1)$-hypercube, or it is This gives the combinatorial construction

   $$\Xi = 1 + z \times (2 \times \Xi + u \times \Xi),$$

   which gives the OGF equation

   $$\Xi(z,u) = 1 + z(2+u)\Xi(z,u).$$

   Therefore, we have the explicit formula

   $$\Xi(z,u) = \frac{1}{1 - z(2+u)}.$$

2. The number of k-faces of the unit N-hypercube is

   $$\begin{aligned}[u^k][z^N]\Xi(z,u) &= [u^k][z^N]\frac{1}{1-z(2+u)} \\ &= [u^k](2+u)^N \\ &= 2^{N-k}\binom{N}{k}.\end{aligned}$$

3. The enumeration OGF is

   $$\Xi(z,1) = \frac{1}{1-z(2+1)} = \frac{1}{1-3z} = \sum_{N=0}^{\infty} 3^N z^N,$$

   and the accumulated cost OGF is

   $$\Xi_u(z,1) = \frac{z}{(1-z(2+1))^2} = \frac{z}{(1-3z)^2} = \sum_{N=0}^{\infty} N3^{N-1}z^N.$$

   Therefore, the expected value of the dimension of a random face of the unit N-hypercube is

   $$\frac{[z^N]\Xi(z,1)}{[z^N]\Xi_u(z,1)} = \frac{N3^{N-1}}{3^N} = \frac{N}{3}.$$