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COS 488 Problem Set #8 Question #1

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We will show the general identity, since this suffices to show the first part of the question. We note that the average for size-n Cayley trees is given by $\frac{[z^n]T_u(z,1)}{[z^n]T(z,1)}$ where $T(z,u)$ is the bivariate EGF for the number of nodes of outdegree k. The combinatorial construction for $\mathcal{T}$ is as follows:

$$\mathcal{T} = \mathcal{Z} \times SET(\mathcal{T}) - \mathcal{Z} \times SET_k(\mathcal{T}) + \mathcal{Z} \times u \times SET_k(\mathcal{T})$$

$$T(z,u) = ze^{T(z,u)} + z(u-1)\frac{T(z,u)^k}{k!}$$

$$z = \frac{T(z,u)}{e^{T(z,u)} + (u-1)\frac{T(z,u)^k}{k!}}$$

As a result, if $f(y) = \frac{y}{e^y + (u-1)\frac{y^k}{k!}}$, then $f(T(z,u)) = z$ so by Lagrange inversion

$$\begin{aligned}
[z^n]T(z,u) &= \frac{1}{n}[y^{n-1}]\left(e^y + (u-1)\frac{y^k}{k!}\right)^n \\
&= \frac{1}{n}[y^{n-1}]\sum_{i=0}^{n}\binom{n}{i}\left(\frac{u-1}{k!}\right)^i y^{ik}e^{(n-i)y} \\
&= \frac{1}{n}\sum_{i=0}^{\lfloor n/k \rfloor}\binom{n}{i}\left(\frac{u-1}{k!}\right)^i \frac{(n-i)^{n-ik-1}}{(n-ik-1)!} \\
[z^n]T(z,1) &= \frac{1}{n}\binom{n}{0}\frac{n^{n-1}}{(n-1)!} = \frac{n^{n-1}}{n!} \\
[z^n]T_u(z,u) &= \frac{1}{n}\sum_{i=1}^{\lfloor n/k \rfloor}\binom{n}{i}\frac{i(u-1)^{i-1}}{k!^i}\frac{(n-i)^{n-ik-1}}{(n-ik-1)!} \\
[z^n]T_u(z,1) &= \frac{1}{n}\binom{n}{1}\frac{1}{k!}\frac{(n-1)^{n-k-1}}{(n-k-1)!} = \binom{n-1}{k}\frac{(n-1)^{n-k-1}}{(n-1)!}
\end{aligned}$$

By Stirling's approximation,

$$\binom{n-1}{k} \approx \frac{\sqrt{2\pi(n-1)}((n-1)/e)^{n-1}}{k!\sqrt{2\pi(n-k-1)}((n-k-1)/e)^{n-k-1}} \approx \frac{e^k(n-1)^k\left(1-\frac{k}{n-k-1}\right)^{n-k-1}}{k!} \approx \frac{(n-1)^k}{k!}$$

As a result, $[z^n]T_u(z,1) \approx \frac{(n-1)^{n-1}}{k!(n-1)!}$. Our expression for the average then yields

$$\frac{[z^n]T_u(z,1)}{[z^n]T_u(z)} \approx \frac{n}{k!}\left(1-\frac{1}{n}\right)^{n-1} \approx \frac{n}{k!}e^{-1}$$