Analytic Combinatorics Note IV.28

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-1 Drawings?

Let $f(z)\ln\left(\frac{1}{1-\ln\frac{1}{1-z}}\right)$ be the generating function for supernecklaces. We have

$$f'(z) = \left(1-\ln\frac{1}{1-z}\right)\cdot\frac{(1-z)\frac{1}{(1-z)^2}}{\left(1-\ln\frac{1}{1-z}\right)^2} = \frac{1}{(1-z)\left(1-\ln\frac{1}{1-z}\right)} = \frac{1}{(1-z)(1+\ln(1-z))}.$$

Write $f'(z) = \frac{k(z)}{g(z)}$ where $k(z) = \frac{1}{1-z}$ and $g(z) = 1+\ln(1-z)$. Note that f' has two poles: one at $z = 1$ and one when $\ln(1-z) = -1$, i.e. $z = 1-e^{-1}$. The second pole is closer to the origin; call it α. Note that α is a pole of order 1 since $g'(\alpha) = \frac{-1}{1-\alpha} \neq 0$. By the theorem on Slide 59, we have $[z^N]f'(z) \sim c\alpha^{-N}$, where $c = (-1)\frac{\frac{1}{1-\alpha}}{\alpha g'(\alpha)} = \frac{1}{\alpha}$. Thus we have $[z^N]f'(z) \sim \alpha^{-(N+1)}$.

We have $[z^N]f(z) = \frac{1}{N}[z^{N-1}]f'(z) \sim \frac{1}{N}\alpha^{-N}$. Thus we have $[z^n]f(z) \sim \frac{1}{n}(1-e^{-1})^{-n}$. Since $f(z)$ is an EGF, to estimate the number of supernecklaces of size n we multiply by $n!$. Thus, the number of supernecklaces of size n is asymptotically $\boxed{(n-1)!(1-e^{-1})^{-n}}$. We can also use Stirling's formula on $(n-1)!$, giving us that the number of supernecklaces is asymptotically

$$\frac{1}{n}\sqrt{2\pi n}\left(\frac{n}{e}\right)^n(1-e^{-1})^{-n} = \boxed{\sqrt{\frac{2\pi}{n}}\left(\frac{n}{e-1}\right)^n}.$$