Analytic Combinatorics Program IV.1

Eric Neyman
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I wrote the following code:

```
import java.util.Arrays;

public class Permutations {
    public static void main(String[] args) {
        final int arraySize = 20;
        final int sampleSize = 100000000;
        int counter = 0;
        for (int i = 0; i < sampleSize; i++) {
            int[] perm = fisherYates(arraySize);
            boolean validPerm = true;
            for (int j = 0; j < arraySize; j++) {
                if (perm[perm[j]] == j) {
                    validPerm = false;
                    break;
                }
            }
            if (validPerm) {
                counter++;
            }
        }
        System.out.println((double) counter / sampleSize);
    }

    public static int[] fisherYates(int arraySize) {
        // Generates a random permutation of size arraySize.
        int[] perm = new int[arraySize];
        for (int i = 0; i < arraySize; i++) {
            perm[i] = i;
        }
        for (int i = arraySize - 1; i > 0; i--) {
            int rand = (int) (Math.random() * (i + 1));
            int temp = perm[rand];
            perm[rand] = perm[i];
            perm[i] = temp;
        }
        return perm;
    }
}
```

For $N = 10$ (i.e. arraySize equal to 10), we obtain an fraction of 0.22315024 from a sample size of one hundred million. (The standard deviation here is on the order of one divided by the square root of $100,000,000$, so this is almost certainly accurate to within 0.001.) For $N = 20$, we obtain a fraction of 0.22314142.

A permutation with no singleton or doubleton cycles is a set of cycles of size greater than 2, i.e. $P = SET(CYC_{>2}(Z))$. We thus obtain the EGF

$$P(z) = \ln \frac{1}{1-z} - z - \frac{1}{2}z^2 = \frac{1}{1-z} \cdot e^{-z-\frac{1}{2}z^2}$$

We have written $P(z)$ as a convolution of a function with $1 + z + z^2 + \ldots$, so the N-th coefficient of $P(z)$ is the sum of the first N coefficients of $e^{-z-\frac{1}{2}z^2}$. We thus have

$$\lim_{N\to\infty} [z^N]P(z) = \sum_{n=0}^{\infty} e^{-z-\frac{1}{2}z^2} = e^{-z-\frac{1}{2}z^2} \,|_{z=1} = e^{-1.5} \approx 0.22313016.$$

This is precisely the value we want: being an EGF, the N-th coefficient of $P(z)$ is already divided by $N!$, so we do not need to divide by $N!$ to compute the ratio. Our simulation indeed returns values very close to this limit, with the value returned for $N = 20$ slightly closer than the value returned for $N = 10$.