Analytic Combinatorics Homework 9 Question and Answer

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Let $f(z) = \frac{3z^2-4z+1}{z^3-2z^2+z-2}$.

(a) Write the partial fraction decomposition of $f(z)$.

(b) What is the radius of convergence of f?

(c) Expand $f(z)$ as a series around $z = 0$; write down the first four terms of the series. Describe how the coefficients behave.

(a) We have $z^3 - 2z^2 + z - 2 = (z^2+1)(z-2) = (z+i)(z-i)(z-2)$. We thus write

$$\begin{aligned} f(z) &= \frac{A}{z+i} + \frac{B}{z-i} + \frac{C}{z-2} = \frac{A(z-i)(z-2) + B(z+i)(z-2) + C(z+i)(z-i)}{(z+i)(z-i)(z-2)} \\ &= \frac{(A+B+C)z^2 + ((-2-i)A + (-2+i)B)z + (2iA - 2iB + C)}{(z+i)(z-i)(z-2)}. \end{aligned}$$

We thus have $A+B+C=3$, $(-2-i)A + (-2+i)B = -4$, and $2iA - 2iB + C = 1$. Solving, we get $A = B = C = 1$, so

$$f(z) = \frac{1}{z+i} + \frac{1}{z-i} + \frac{1}{z-2}.$$

(b) The poles of smallest magnitude have magnitude 1, so the radius of convergence is 1.

(c) We have

$$\begin{aligned} \frac{1}{z+i} &= \frac{-i}{1-iz} = i(-1 - iz - (iz)^2 - (iz)^3 + \dots) \\ \frac{1}{z-i} &= \frac{i}{1+iz} = i(1 - iz + (iz)^2 - (iz)^3 + \dots) \\ \frac{1}{z-2} &= \frac{-1}{2} \cdot \frac{1}{1 - \frac{1}{2}z} = \frac{-1}{2}\left(1 + \frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} + \dots\right). \end{aligned}$$

Adding the first two terms, we get $2z - 2z^3 + 2z^5 - \dots$. Thus, we have

$$f(z) = \frac{-1}{2} + \frac{7}{4}z - \frac{1}{8}z^2 - \frac{33}{16}z^3 + \dots.$$

The even coefficients approach 0. The 1 (mod 4) coefficients approach 2. The 3 (mod 4) coefficients apporach -2.