COS 488: AC week 3 QA1

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Prove the following identity:

$$\int_0^{2\pi} \cosh(\cos x)\cos(\sin x)dx = 2\pi$$

(Hint: try using an expression using $\cosh(z)$ having residue 1.)

Disclaimer: My apologies if you wanted a more applied question. Since the week's material was only on complex analysis without applications except through derivation of a theorem we already knew without proof, I thought this would add some variety, especially since it uses most of the results given in class in a straightforward manner.

Solution:
As per the hint, we construct (the most simple) [an] expression using $\cosh(z)$ and having residue 1, namely:

$$h(z) = \frac{\cosh z}{z}$$

By the residue theorem/lemma (slide 34 of AC04-Poles.pdf) we have that

$$\int_\lambda h(z)dz = 2\pi i * 1 = 2\pi i$$

But we may also integrate it explicitly using the methods shown. We will integrate along a unit circle in the complex plane (which we are allowed to do, by the theorem by Cauchy given in lecture), make a change of variables to theta, then split the integral into imaginary and real terms.

$$\int_\lambda \frac{\cosh z}{z} dz = 1/2 \int_\lambda \frac{e^z + e^{-z}}{z} dz$$

let $z = e^{i\theta}$ and thus $dz = ie^{i\theta}d\theta$, then we have the following substitution:

$$= 1/2 \int_0^{2\pi} \frac{e^{\cos\theta + i\sin\theta} + e^{-\cos\theta - i\sin\theta}}{e^{i\theta}} ie^{i\theta}d\theta = i/2 \int_0^{2\pi} e^{\cos\theta}e^{i\sin\theta} + e^{-\cos\theta}e^{-i\sin\theta}d\theta$$

now recall Euler's formula (proven on slide 20 of AC04-Poles.pdf) and apply it

$$= 1/2 \int_0^{2\pi} i(e^{\cos\theta}\cos(\sin\theta) + e^{-\cos\theta}\cos(\sin\theta)) - (e^{\cos\theta}\sin^2\theta - e^{-\cos\theta}\sin^2\theta)d\theta$$

$$= i\int_0^{2\pi} \cosh(\cos\theta)\cos(\sin\theta)d\theta - \int_0^{2\pi} \sinh(\cos\theta)\sin^2\theta d\theta = 2\pi i$$

(where the final equality is from our residue identity)
Thus we see the sinh integral is 0, and the other is 2π, as desired.
Verified with Mathematica.
Notes: This is a very cool identity I stumbled across, even if it may not be very useful in practice since it's so specific.
I know this question is a bit computation-intensive, but it really only uses things taught in our lecture (as cited), and is fairly straightforward with the hint. Maybe I could make the hint a little more exact and just give the expression to integrate, depending on the students' time constraints.