Homework 9: Note III.17

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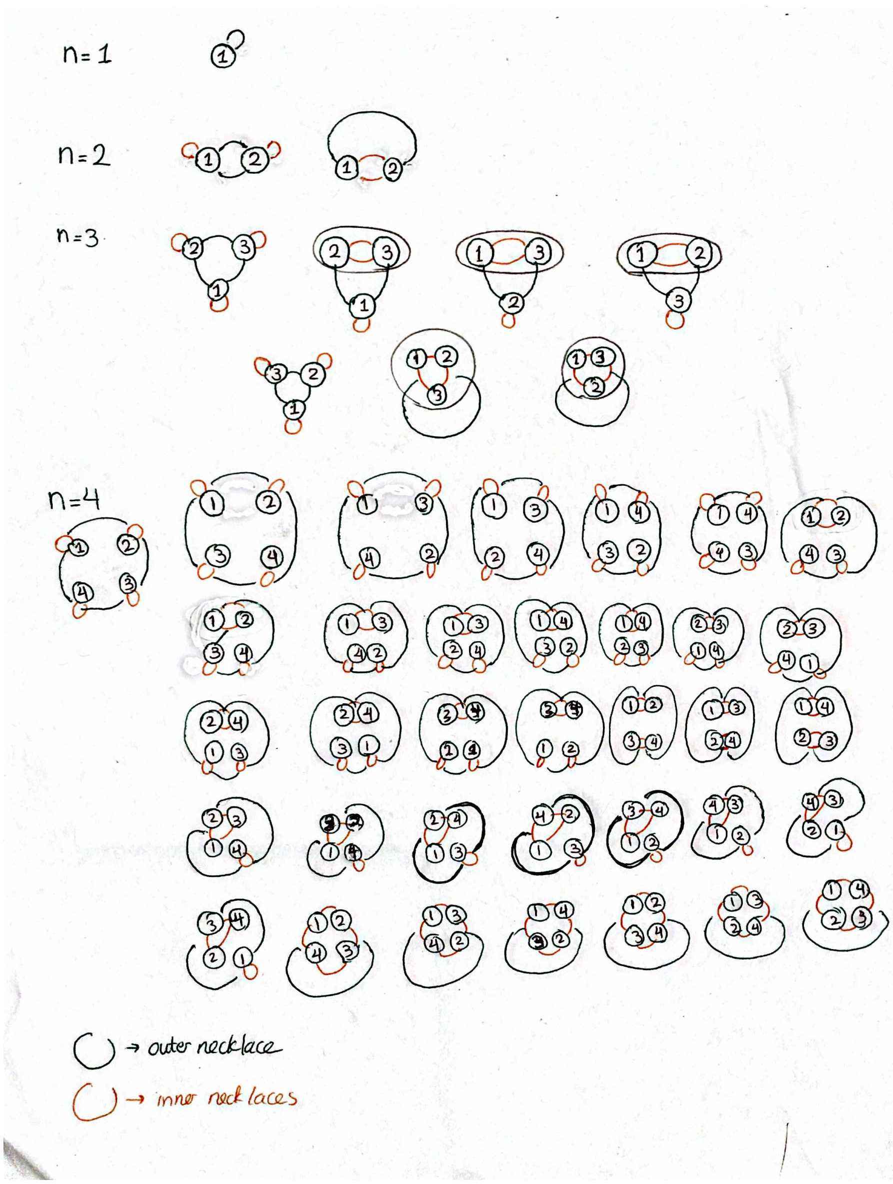
All the necklaces of length $n = 1, 2, 3, 4$ are drawn below.



The generating function for supernecklaces of the third type is

$$S(z) = \ln\left(\frac{1}{1-\ln\left(\frac{1}{1-z}\right)}\right)$$
$$\frac{\partial}{\partial z}S(z) = \frac{1}{(1-z)(1-\ln\left(\frac{1}{1-z}\right))} = \frac{f(z)}{g(z)}.$$

Since the derivative is a meromorphic function, we can apply the meromorphic function transfer theorem to derive asymptotics.

The roots of the denominator are 1 and $1-\frac{1}{e}$, so the pole of smallest modulus is $\rho = 1-\frac{1}{e}$ with multiplicity 1 and order $M = 1$. Applying the theorem, we have

$$c = (-1)^1 \frac{1 \cdot 1}{\left(1-\frac{1}{e}\right)^1 \frac{\partial}{\partial z}(1-z)(1-\ln\left(\frac{1}{1-z}\right))|_{z=1-\frac{1}{e}}}$$
$$= \frac{-1}{(1-\frac{1}{e})\cdot(-1)} = \left(1-\frac{1}{e}\right)^{-1}$$

and

$$[z^n]\frac{\partial}{\partial z}S(z) = \left(1-\frac{1}{e}\right)^{-1}\left(\frac{1}{1-\frac{1}{e}}\right)^n n^{1-1} = \left(\frac{1}{1-\frac{1}{e}}\right)^{n-1}$$
$$[z^n]S(z) = n[z^{n+1}] = n\left(\frac{1}{1-\frac{1}{e}}\right)^n = n(1-e^{-1})^{-1},$$

as desired.