Homework 9: Question and Answer

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Complex Differentiation and Integration Warm-up

1. Verify that $\frac{d}{dt}e^{zt} = ze^{zt}$ for $z \in \mathbb{C}$.

2. Evaluate following integral when C is a circle of radius 1, 3, or 5 centered at the origin:

$$\int_C \frac{z^2 + z + 1}{z^2 + 2z - 8} dz.$$

Answer

1. Let $z = a + ib$. Then

$$\begin{aligned} e^{zt} &= e^{(a+ib)t} = e^{at} \cdot e^{ibt} \\ \frac{d}{dz} e^{zt} &= e^{at} \frac{d}{dt}(e^{ibt}) + a e^{at} e^{ibt}. \end{aligned}$$

   Note that

$$\begin{aligned} \frac{d}{dt}(e^{ibt}) &= \frac{d}{dt}(\cos(bt) + i\sin(bt)) \\ &= -b\sin(bt) + ib\cos(bt) \\ &= ib(\cos(bt) + i\sin(bt)) \\ &= ibe^{ibt}. \end{aligned}$$

   Plugging back into the original equation, we have

$$\begin{aligned} \frac{d}{dz} e^{zt} &= e^{at} ibe^{ibt} + ae^{at}e^{ibt} \\ \frac{d}{dz} e^{zt} &= e^{at}e^{ibt}(a+ib) \\ \frac{d}{dz} e^{zt} &= ze^{zt}. \end{aligned}$$

2. Note that the denominator of the integrand can be factored into $(z+4)(z-2)$, so the function has singularities at $z = 2$ and $z = -4$.

   Let C_1 be a circle of radius 1 circled at the origin. Since C_1 does not contain any of the singularities of the integrand, the integrand is analytic inside C_1. By Cauchy's theorem,

$$\int_{C_1} \frac{z^2 + z + 1}{z^2 + 2z - 8} dz = \boxed{0}.$$

   Next, let C_3 be a circle of radius 3 circled at the origin, and note that C_3 contains exactly one singularity of the integrand at $z = 2$. We can rewrite the integral as

$$\int_{C_3} \frac{z^2 + z + 1}{z^2 + 2z - 8} dz = \int_{C_3} \frac{\frac{z^2+z+1}{(z+4)}}{z - 2} dz$$

   where the function in the numerator is analytic in C_3 and Cauchy's integral formula applies directly:

$$\int_{C_3} \frac{\frac{z^2+z+1}{(z+4)}}{z-2} dz = 2\pi i \frac{z^2 + z + 1}{(z+4)}\bigg|_{z=2} = \pi i \cdot \frac{7}{6} = \boxed{\frac{7}{3}\pi i}.$$

Finally, let C_5 be a circle of radius 5 circled at the origin, and note that C_5 contains both singularities of the integrand at $z = 2$ and $z = -4$. Using the residue theorem, we have

$$\begin{aligned}
\int_{C_5} \frac{z^2+z+1}{z^2+2z-8} dz &= 2\pi i \sum_{s \in \{2,-4\}} \text{Res}_{z=s} \left(\frac{z^2+z+1}{z^2+2z-8} \right) \\
&= 2\pi i \left(\text{Res}_{z=2} \left(\frac{\frac{z^2+z+1}{z+4}}{z-2} \right) + \text{Res}_{z=-4} \left(\frac{\frac{z^2+z+1}{z-2}}{z+4} \right) \right) \\
&= 2\pi i \left(\left. \frac{z^2+z+1}{z+4} \right|_{z=2} + \left. \frac{z^2+z+1}{z-2} \right|_{z=-4} \right) \\
&= 2\pi i \left(\frac{7}{6} + \frac{13}{-6} \right) \\
&= \boxed{-2\pi i}.
\end{aligned}$$