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COS 488/MAT 474
Problem Set 9, Q1

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AC Note IV.28 *Supernecklaces.* A "supernecklace" of the 3rd type is a labelled cycle of cycles. Draw all the supernecklaces of the 3rd type of size n for $n = 1, 2, 3$, and 4. Then develop an asymptotic estimate of the number of supernecklaces of size n by showing that

$$[z^n] \ln\left(\frac{1}{1 - \ln \frac{1}{1-z}}\right) \sim \frac{1}{n}(1 - e^{-1})^{-n}.$$

Solution. The EGF for cycles is $\ln \frac{1}{1-z}$, so the description of a supernecklace as a "cycle of cycles" immediately yields the EGF

$$S(z) = \ln\left(\frac{1}{1 - \ln \frac{1}{1-z}}\right).$$

The power series expansion for this function begins

$$S(z) = z + z^2 + \frac{7}{6}z^3 + \frac{35}{24}z^4 + \cdots,$$

which means there is 1 supernecklace of size 1, $1 \cdot 2! = 2$ supernecklaces of size 2, $\frac{7}{6} \cdot 3! = 7$ supernecklaces of size 3, and $\frac{35}{24} \cdot 4! = 35$ supernecklaces of size 4.

I have drawn all the supernecklaces of size $n = 1, 2$, and 3. The "outer" cycles are in red, and the "inner" cycles are in blue.

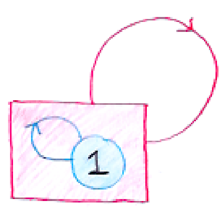

Figure 1: The 1 supernecklace of size $n = 1$ ($S_1 = 1$).

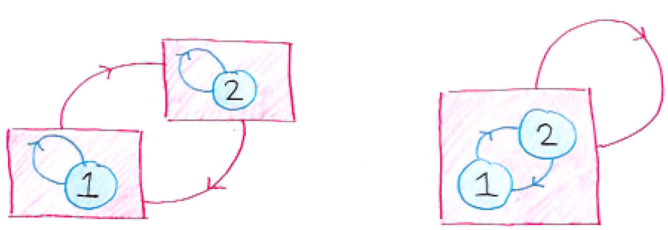

Figure 2: The 2 supernecklaces of size $n = 2$ ($S_2 = 2$).

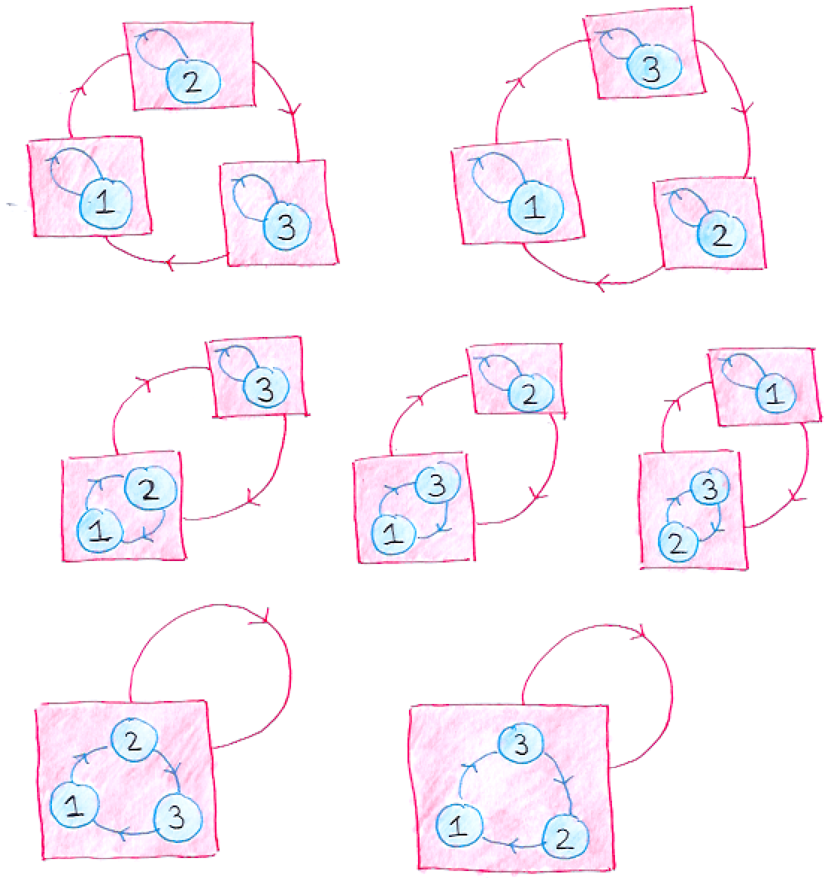

Figure 3: The 7 supernecklaces of size $n = 3$ ($S_1 = 1$).

For $n = 4$, there are too many supernecklaces to draw them all, so I will describe them. A supernecklace of size 4 can have either 1, 2, 3, or 4 outer ("red") beads.

- For supernecklaces with 4 red beads, each red bead contains a blue 1-cycle, so there are 6 of them, corresponding to the $3! = 6$ ways to order the 4-cycle.

- For supernecklaces with 3 red beads, one red bead contains a blue 2-cycle and the other two red beads each contain a blue 1-cycle (There are $\binom{4}{2} = 6$ ways to choose the 2-cycle, and 2 ways to order the 3 red beads, so there are $6 \cdot 2 = 12$ of these).

- For supernecklaces with 2 red beads, there are two cases: either each red bead contains a blue 2-cycle (there are 3 ways to choose two pairs of 2, so there are 3 of these), or one red bead contains a blue 3-cycle and the other contains a blue 1-cycle (there are 4 ways to choose a singleton and 2 ways to reorder each 3-cycle, so there are $4 \cdot 2 = 8$ of these).

- For supernecklaces with 1 red bead, the red bead contains a blue 4-cycle, and there are $3! = 6$ ways to order this 4-cycle.

In total, this makes $S_4 = 6 + 12 + 3 + 8 + 6 = 35$ supernecklaces of size $n = 4$, as expected.

Now we'll find an asymptotic estimate of the number of supernecklaces for large n. Taking the derivative of the EGF $S(z)$, we get

$$S'(z) = \frac{1}{(1-z)\left(1 - \ln\frac{1}{1-z}\right)} = \frac{1}{g(z)}.$$

$S'(z)$ is meromorphic, with two poles located at $z = 1$ and $z = 1 - \frac{1}{e}$, each of order 1. Thus, by the transfer theorem for meromorphic functions, we have

$$[z^n]S'(z) \sim c\beta^n,$$

where $\alpha = 1 - \frac{1}{e}$ (the pole of smallest modulus), $\beta = \frac{1}{\alpha}$, and

$$c = \frac{-1}{\alpha \cdot g'(\alpha)} = \frac{-1}{\alpha\left(\ln\frac{1}{1-\alpha} - 2\right)} = \frac{1}{\alpha} = \beta.$$

Therefore,

$$[z^n]S'(z) \sim \beta^{n+1} = \left(1 - \frac{1}{e}\right)^{-(n+1)}.$$

Now we have asymptotic approximations for the coefficients in the power series expansion of $S'(z)$. If we integrate this power series, we get an approximation for the coefficients of $S(z)$:

$$[z^n]S(z) = \frac{1}{n}[z^{n-1}]S'(z) \sim \frac{1}{n}\left(1 - \frac{1}{e}\right)^{-n},$$

as we wanted. Therefore, the number of supernecklaces of size n is asymptotic to

$$S_n \sim \frac{(n-1)!}{\left(1 - \frac{1}{e}\right)^n}.$$