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COS 488/MAT 474
Problem Set 9, Q2

6/5

AC Program IV.1 Compute the percentage of permutations having no singleton or doubleton cycles and compare with the asymptotic estimate from analytic combinatorics, for $N = 10$ and $N = 20$.

Solution.

Computational approach: When choosing a permutation of N atoms with no 1- or 2-cycles, we can first pick a cycle of size $(k+1) \geq 3$ that contains the atom labelled "N". Then, we pick a permutation of the remaining $N - k - 1$ atoms that has no 1- or 2- cycles. k can range from 2 to $N - 1$. The number of ways to choose k atoms to join atom "N" in a cycle is $\binom{N-1}{k}$, and the number of ways to order these $k+1$ atoms in a cycle is $k!$. Therefore, the number of permutations of N atoms containing no 1- or 2- cycles can be computed recursively by the formula

$$P_N = \sum_{k=2}^{N-1} \binom{N-1}{k} \cdot k! \cdot P_{N-k-1} \qquad \text{for } N \geq 3, \text{ with } P_0 = 1, P_1 = 0, P_2 = 0.$$
$$= \sum_{k=2}^{N-1} \frac{(N-1)!}{(N-k-1)!} \cdot P_{N-k-1}.$$

Excellent! +1

I used this formula to write a recursive program to compute the P_N. My Java program is submitted under "Additional files" as Permutation.java. For $N = 10$ and $N = 20$, the numbers of permutations with no 1- or 2- cycles are

```
> java Permutation 10
809856
> java Permutation 20
542853814536802656
```

As percentages of the total number of permutations, we have

$$\frac{809856}{10!} \doteq 0.223174603$$
$$\frac{542853814536802656}{20!} \doteq 0.223130160.$$

Analytic combinatorics approach: The class of permutations with no 1- or 2-cycles is the class of sets of cycles of size ≥ 3, or in symbols,

$$\mathcal{P} = SET(CYC_{\geq 3}(Z)).$$

Translating this to an EGF, we have

$$P(z) = \exp\left(\ln\frac{1}{1-z} - \left(z + \frac{z^2}{2}\right)\right)$$
$$= \frac{1}{(1-z)e^{z+z^2/2}} = \frac{1}{g(z)}.$$

This function has one pole of order 1, located at $z = 1$. So by the transfer theorem for meromorphic functions, the coefficients $\frac{P_N}{N!}$ are asymptotic to a constant c, where

$$c = \frac{-1}{g'(1)} = \left.\frac{1}{z^2e^{z+z^2/2}}\right|_{z=1} = e^{-3/2}.$$

Therefore,

$$P_N \sim e^{-3/2} \cdot N!$$

and the percentage of permutations with no 1- or 2- cycles is asymptotic to

$$\frac{P_N}{N!} \sim e^{-3/2} \doteq 0.223130160.$$

This is a very accurate approximation for large N, as shown by our previous calculations for $N = 10$ and $N = 20$. For $N = 10$, the approximation is accurate to within 10^{-4}, and for $N = 20$, it is accurate to within 10^{-9}.