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COS 488/MAT 474
Problem Set 9, Q&A
Meromorphic Asymptotics

Question The class of supernecklaces of the 1st type is given by the construction

$$\mathcal{S} = CYC(SEQ_{>0}(\mathcal{Z})).$$

Using the transfer theorem for meromorphic functions, show that

$$[z^N]S(z) = \frac{2^N}{N}$$

and thus $S_N \sim 2^N(N-1)!$.

In fact, the number of supernecklaces of the 1st type of size N is exactly equal to

$$S_N = (2^N - 1)(N-1)!.$$

Give a combinatorial justification of this.

Solution.

$$\mathcal{S} = CYC(SEQ_{>0}(\mathcal{Z}))$$
$$\Rightarrow S(z) = \ln \frac{1}{1 - \frac{z}{1-z}} = \ln \frac{1-z}{1-2z}.$$

Take the derivative of both sides:

$$S'(z) = \frac{1}{(1-z)(1-2z)}.$$

This is a meromorphic function whose unique pole of smallest modulus is located at $z = \frac{1}{2}$, and this pole has multiplicity 1. Plugging into the transfer theorem for meromorphic functions, we have

$$\alpha = \frac{1}{2}, \qquad \beta = 2, \qquad c = \frac{-1}{\alpha(4\alpha - 3)} = 2.$$

Therefore,

$$[z^N]S'(z) \sim 2^{N+1}$$
$$\Rightarrow [z^N]S(z) = \frac{1}{N}[z^{N-1}]S'(z) \sim \frac{2^N}{N}$$
$$\Rightarrow S_N \sim 2^N (N-1)!.$$

A supernecklace of the 1st type can be thought as a labelled cycle of atoms, partitioned into sequences of adjacent atoms. There are $(N-1)!$ ways to order an N-cycle. To partition the cycle into sequences, we can think of drawing slashes through some of the spaces between adjacent atoms. These slashes delineate where one sequence ends and the next begins. There are N spaces, and we need to choose at least one of them to put a slash through. There are $2^N - 1$ ways to choose ≥ 1 element from an N-element set. Therefore, the total number of supernecklaces of size N is exactly

$$S_N = (2^N - 1)(N-1)!.$$