COS 488 - Homework 9 - Note IV.28

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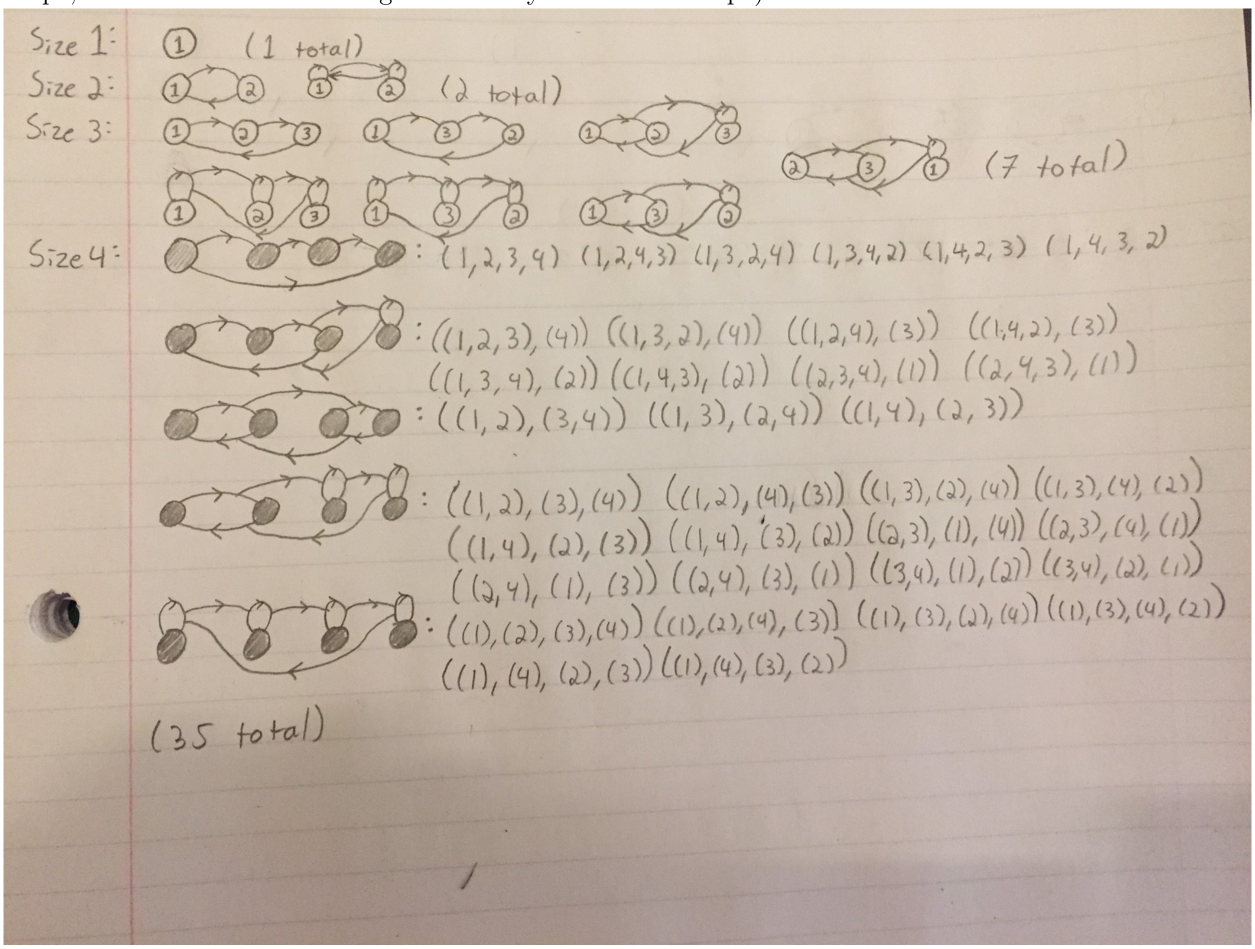
The "supernecklaces" of the third type of sizes 1, 2, 3, and 4 are drawn below (for size 4, I drew each possible shape, and then listed the distinguishable ways to fill that shape):



Let S be the combinatorial class of "supernecklaces" of the third type. Since a "supernecklace" of the third type is a labelled cycle of cycles, we have the construction

$$S = CYC(CYC(Z)),$$

which gives the EGF

$$S(z) = \ln\left(\frac{1}{1-\ln\left(\frac{1}{1-z}\right)}\right).$$

Then,

$$S'(z) = \frac{1}{1-z}\frac{1}{1+\ln(1-z)},$$

which is meromorphic at all points $|z| \le 2$ and analytic at 0 and at all points $|z| = 2$. Since $1-e^{-1}$ is the unique closest pole to the origin of $S'(z)$, and this pole has order 1, we have by the transfer theorem shown in class with $S(z) = \frac{f(z)}{g(z)} = \frac{1/(1-z)}{1+\ln(1-z)}$ that

$$\begin{aligned}
[z^n]S'(z) &\sim \frac{-f(1-e^{-1})}{(1-e^{-1})g'(1-e^{-1})}\left(\frac{1}{1-e^{-1}}\right)^n \\
&= \frac{-e}{(1-e^{-1})(-e)}\left(\frac{1}{1-e^{-1}}\right)^n \\
&= \left(\frac{1}{1-e^{-1}}\right)^{n+1}
\end{aligned}$$

Therefore,

$$[z^n]S(z) = \frac{1}{n}[z^{n-1}]S'(z) \sim \frac{1}{n}(1-e^{-1})^{-n},$$

as desired.
The number of "supernecklaces" of the third type of size n is therefore

$$\begin{aligned}
n![z^n]S(z) &\sim n!\frac{1}{n}(1-e^{-1})^{-n} \\
&\sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n (1-e^{-1})^{-n} \\
&\sim \sqrt{\frac{2\pi}{n}}\left(\frac{n}{e-1}\right)^n.
\end{aligned}$$