COS 488 - Homework 9 - Note IV.28

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The "supernecklaces" of the third type of sizes 1, 2, 3, and 4 are drawn below (for size 4, I drew each possible shape, and then listed the distinguishable ways to fill that shape):

Size 1: (1 total) Size 2: B a 2 (2 total) Size 3: 2 (7 total) a.C. ,3,4) (1,2,4,3) (1,3,2,4) (1,3,4,2) (1,4,2,3) (1,4,3,2) Size 4: : ((1,2,3), (4)) ((1,3,2), (4)) ((1,2,4), (3)) ((1,4,2), (3))((1,3,4), (2)) ((1,4,3), (2)) ((2,3,4), (1)) ((2,4,3), (1)): ((1,2), (3,4)) ((1,3), (2,4)) ((1,4), (2,3)) $\begin{pmatrix} (1, \lambda), (3), (4) \end{pmatrix} ((1, \lambda), (4), (3) \end{pmatrix} ((1, 3), (\lambda), (4) \end{pmatrix} ((1, 3), (4), (\lambda) \end{pmatrix}$ $((1, 4), (\lambda), (3)) ((1, 4), (3), (\lambda)) ((3, 3), (1), (4)) ((3, 3), (4), (1))$ $((1, 4), (1), (3)) ((2, 4), (3), (1)) ((3, 4), (1), (\lambda)) ((3, 4), (\lambda), (\lambda))$ $((1), (\lambda), (3), (4)) ((1), (\lambda), (4), (3)) ((1), (3), (\lambda), (4)) ((1), (3), (4), (2))$ $((1), (\lambda), (3), (4)) ((1), (4), (3), (\lambda))$ $((1), (4), (\lambda), (3)) ((1), (4), (3), (\lambda))$ (35 total)

Let S be the combinatorial class of "supernecklaces" of the third type. Since a "supernecklace" of the third type is a labelled cycle of cycles, we have the construction

$$S = CYC(CYC(Z)),$$

which gives the EGF

$$S(z) = \ln\left(\frac{1}{1 - \ln\left(\frac{1}{1-z}\right)}\right).$$

Then,

$$S'(z) = \frac{1}{1-z} \frac{1}{1+\ln(1-z)},$$

which is meromorphic at all points $|z| \le 2$ and analytic at 0 and at all points |z| = 2. Since $1 - e^{-1}$ is the unique closest pole to the origin of S'(z), and this pole has order 1, we have by the transfer theorem shown in class with $S(z) = \frac{f(z)}{g(z)} = \frac{1/(1-z)}{1+\ln(1-z)}$ that

$$[z^{n}]S'(z) \sim \frac{-f(1-e^{-1})}{(1-e^{-1})g'(1-e^{-1})} \left(\frac{1}{1-e^{-1}}\right)^{n}$$
$$= \frac{-e}{(1-e^{-1})(-e)} \left(\frac{1}{1-e^{-1}}\right)^{n}$$
$$= \left(\frac{1}{1-e^{-1}}\right)^{n+1}$$

Therefore,

$$[z^{n}]S(z) = \frac{1}{n}[z^{n-1}]S'(z) \sim \frac{1}{n}(1-e^{-1})^{-n},$$

as desired.

The number of "supernecklaces" of the third type of size n is therefore

$$n![z^{n}]S(z) \sim n!\frac{1}{n}(1-e^{-1})^{-n}$$
$$\sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^{n} (1-e^{-1})^{-n}$$
$$\sim \sqrt{\frac{2\pi}{n}} \left(\frac{n}{e-1}\right)^{n}.$$