COS 488 - Homework 9 - Question & Answer

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1 Question

1. Let $f(z)$ be an analytic function on the closed unit disk, and let γ denote the unit circle. Show that

$$\oint_\gamma \frac{f(z)}{z} dz = 2\pi i f(0).$$

2. Under the same assumptions as in part 1, show that

$$\int_0^{2\pi} f(e^{i\theta}) d\theta = 2\pi f(0).$$

3. Evaluate

$$\int_0^{2\pi} \cosh(\cos(\theta)) \cos(\sin(\theta)) d\theta.$$

   Hint: Let $f(z) = \cosh(z) = \frac{e^z + e^{-z}}{2}$ and consider $\Re(f(z))$.

   Note: Wolfram Alpha cannot compute a closed form for this integral!

2 Answer

1. From Cauchy's coefficient formula, we have

$$\frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{z^{n+1}}dz = [z^n]f(z).$$

Specializing to $n = 0$, we have

$$\frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{z}dz = [z^0]f(z) = f(0).$$

2. Let $z = e^{i\theta}$, so that $dz = ie^{i\theta}d\theta$. Then, under this change of variables, the integral from part 1 becomes

$$\frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{z}dz = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(e^{i\theta})}{e^{i\theta}}ie^{i\theta}d\theta = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})d\theta,$$

so

$$\int_0^{2\pi} f(e^{i\theta})d\theta = 2\pi f(0).$$

3. As suggested, we let $f(z) = \cosh(z)$, so that

$$\begin{aligned}
\Re\left(f(e^{i\theta})\right) &= \Re\left(\frac{e^{\cos(\theta)+i\sin(\theta)} + e^{-\cos(\theta)-i\sin(\theta)}}{2}\right)\\
&= \Re\left(\frac{e^{\cos(\theta)+i\sin(\theta)} + e^{-\cos(\theta)-i\sin(\theta)}}{2}\right)\\
&= \frac{e^{\cos(\theta)}\cos(\sin(\theta)) + e^{-\cos(\theta)}\cos(-\sin(\theta))}{2}\\
&= \frac{e^{\cos(\theta)} + e^{-\cos(\theta)}}{2}\cos(\sin(\theta))\\
&= \cosh(\cos(\theta))\cos(\sin(\theta)).
\end{aligned}$$

Then, from part 2, we have

$$\begin{aligned}
\int_0^{2\pi} \cosh(\cos(\theta))\cos(\sin(\theta))d\theta &= \int_0^{2\pi} \Re\left(f(e^{i\theta})\right)d\theta\\
&= \Re(2\pi f(0))\\
&= \Re(2\pi \times 1)\\
&= 2\pi.
\end{aligned}$$