(6 + 9 + 7 + 9 + 9 + 7 + 8 + 3) 72.5%

COS 488 Written Exam 1 Spring 2017

There are nine questions on this exam, weighted as indicated at the bottom of this page. There is one question per lecture, numbered corresponding to the lectures, *not in order of difficulty*. If a question seems difficult to you, skip it and come back to it.

Policies. The exam is closed book, though that you are allowed to use a single-page one-sided hand-written cheatsheet. No calculators or other electronic devices are permitted. Give your answers and show your work in the space provided. You will have 80 minutes to complete the test. **This exam will be preprocessed by computer.** If you use pencil (and eraser), write darkly. Fill in circles *completely* when asked. Write all answers inside the designated rectangles. Do not write on the corner marks.

This page. *Print your name and login ID on this page* (now), and write out and sign the Honor Code pledge.

Discussing this exam. As you know, discussing the contents of this exam before solutions have been posted is a serious violation of the Honor Code.

Name Dylan Mavrides

Login Mavrides

"I pledge my honor that I have not violated the Honor Code during this examination."

I pledge my honor that I have not violated the Honor Code during this examination.

[*copy the pledge here*]

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Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	TOTAL
/11	/11	/11	/11	/11	/11	/11	/11	/11	/99

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Q1. Analysis of Algorithms. In a particular (fictitious) sorting application with cloud computing, the cost of sorting files of size less than 10^6 is negligible. Otherwise, the cost of comparisons is such that the budget only can cover 10^{12} comparisons. Of the following, which is the largest file that can be sorted within budget, using the standard Quicksort algorithm (with cutoff to a free sort for files of size less than 10^6)?

total

6 ✓

- 10^9 ○
- 10^{10} ○
- 10^{11} ●
- 10^{12} ○
- 10^{13} ○
- 10^{14} ○

Q2. Recurrences. Match each recurrence with an expression that can make it telescope, by writing the letter to the left of the recurrence in the box to the right of the expression. Find a matching where each letter is used exactly once (and two boxes are left blank).

$(n+1)(n+2)(n+3)$ $\boxed{D}$

$n(n+2)(n+4)$ $\boxed{}$

2^{n+1} $\boxed{B}$

$(n+1)!$ $\boxed{}$

$(n-1)!/4^n$ $\boxed{C}$

$n(n-1)$ $\boxed{A}$

$(n+1)(n+2)$ $\boxed{E}$

$n(n-1)$

A $(n+1)a_{n+1} = (n-2)a_{n-1} + n$

B $a_{n+1} = 4a_{n-1} + (n+1)(n+2)$

C $na_n = 4a_{n-1} + (n+1)(n+2)$

D $na_{n+1} = (n+4)a_{n-1} + n + 4$

E $(n+1)a_{n+1} = (n+2)a_{n-1} + n$

Q3. Generating functions. Match each sequence with its EGF, by writing the letter to the left of the sequence in the box to the right of the EGF.

A 0, −1, 1, −1, 1, −1, . . .

B 0, 1, 3, 7, 15, 31, 63, . . .

C 0, 1, 2, 3, 4, 5, 6, . . .

D 0, 1, 4, 9, 16, 25, 36, . . .

E 0, 1, 2, 6, 24, 120, 720, . . .

factorial shifted

$(e^z - e^{-z})/2$	A
ze^z	C
$e^{2z} - e^z$	B
$(z^2 + z)e^z$	D
$\cancel{z}e^{-z} - 1$	
$\dfrac{z}{1-z}$	F
$e^{z^2} - 1$	

Q4. Asymptotics. Match each expression with the value that most closely approximates it, by writing the letter to the left of the sequence in the box to the right of the value.

A	$e^{.01}$
B	$\ln(1000)/\ln(999)$?
C	$1.001^{694} - 1$
D	$(100 + \ln(100!))/(100\ln(100))$
E	$\ln(2.7183)$

1.001012	$\boxed{\text{D}}$
1.000144	$\boxed{\text{C}}$
1.000023	$\boxed{\text{B}}$
1.010050	$\boxed{\text{A}}$
1.001111	$\boxed{}$
1.000007	$\boxed{\text{E}}$
1.005000	$\boxed{}$

Q5. Analytic Combinatorics. Indicate the combinatorial class corresponding to each construction (if present), by writing the letter to the left of the sequence in the box to the right of the construction. The correct answer may not use all letters, may leave one or more boxes blank, and/or may use one or more letters more than once.

A permutations

B binary strings

C cycles

D derangements

E binary trees

$A = Z + A \times A$	E
$A = SET(CYC_{>1}(Z))$	D
$A = E + Z \star A$	
$A = SEQ(Z_0 + Z_1)$	B
$A = CYC_{>0}(Z)$	C
$A = A \times (Z_0 + Z_1)$ E+	
$A = SEQ(Z)$	A

Q6. Trees. A *binary trie structure* is a binary tree with two types of external nodes (void and nonvoid) with the restriction that void external nodes do not appear in leaves. The number of different binary trie structures with n external nodes is 1, 4, and 17 when n is 2, 3, and 4, respectively. Complete the following derivation to determine an asymptotic approximation to the number of binary trie structures of with n external nodes.

combinatorial construction

$$T = Z_{\blacksquare} + (Z_{\bigcirc} \times T \times T) + 2(Z_{\square} \times Z_{\bigcirc} \times (T - Z_{\blacksquare}))$$

EGF equation

$$T(z) = z + zT(z)^2 + 2(z^2(T(z) - z))$$

explicit form of EGF

$$T(z) = z + zT(z)^2 + 2z^2T(z) - 2z^3$$

$$zT(z)^2 + (2z^2-1)T(z) + (z - 2z^3) = 0$$

$$\frac{1-2z^2 \pm \sqrt{(2z^2-1)^2 - 4z(z-2z^3)}}{2z} = T(z)^2 = \frac{1-2z^2 - \sqrt{(6z^2-1)(2z^2-1)}}{2z}$$

asymptotic approximation

only the last term doesn't go to zero (exponentially small) and can be rewritten as:

$$\frac{-\sqrt{(1-6z^2)(1+\sqrt{2}z)}/2z}{\left(1-z/2^{\frac{1}{2}}\right)^{-\frac{1}{2}}}$$

thus $\rho = \frac{1}{\sqrt{2}}$, $\alpha = -\frac{1}{2}$

$$\sim \frac{f(\rho)}{\Gamma(\alpha)}\rho^{-n}n^{\alpha-1} = \frac{\sqrt{-2}}{2\sqrt{\pi}}\frac{\sqrt{2}^n}{n^{3/2}}$$

Q7. Permutations. Complete the following derivation to determine the number of permutations of size n that have cycles all of even length.

combinatorial construction

$$\left[\text{SET}(\text{CYC}_{\text{even}}(Z)) = \text{SET}((\text{CYC}_2(Z) + \text{CYC}_4(Z) + \dots)) \right.$$

$$P_e * Z = P_{O_2}$$

$$P_{e_2} =$$

EGF equation

$$P(z) = e^{\left(\frac{z^2}{2} + \frac{z^4}{4} + \dots\right)}$$

explicit form of EGF

$$P(z) = e^{\sum_{k \geq 0} \frac{z^{2k}}{2k}}$$

exact value of coefficients

$$\left(\frac{(2n)!}{n! 2^n} \right)^2$$

Q8. Strings. Determine the OGF for the number of strings not containing the pattern 1011.

1011

1011011

combinatorial constructions

$E + (Z_0 + Z_1) \times B = B + P$ ~exactly 1 1011

$Z_{1011} \times B = P + Z_{101} \times P$

OGF equations

$1 + 2zB = B + P$, $z^4 B = P + z^3 P$

$P = \dfrac{z^4 B}{z^3 + 1}$

$1 + 2zB = B + \dfrac{z^4 B}{z^3 + 1}$

$B = \dfrac{1}{\frac{z^4}{z^3+1} - 2z + 1}$

explicit form of OGF

$\cancel{1+2zB}$

$B = \dfrac{1}{\frac{z^4}{z^3+1} - 2z + 1}$

Q9. Words and Maps. This problem involves enumerating the mappings where every character appears exactly twice or not at all. Such mappings must be of even length. There are 2 such mappings of length 2 (11 and 22) and 36 of them of length 4:

1122	1212	1221	2112	2121	2211	1133	1313	1331	3113	3131	3311
1144	1414	1441	4114	4141	4411	2233	2323	2332	3223	3232	3322
2244	2424	2442	4224	4242	4422	3344	3434	3443	4334	4343	4433

By a direct combinatorial argument, it is not difficult to derive the explicit form

$$C_{2n} = \binom{2n}{n}\frac{(2n)!}{2^n}$$

A. Give an asymptotic estimate (~-approximation) of this quantity.

$\frac{4^N}{\sqrt{\pi N^3}}$ ~~$\left(1+O\left(\frac{1}{t}\right)\right)\left(\frac{(N+1)(2n)!}{2^n}\right)$~~ $\frac{(2n!)^2}{(n!)^2 2^n}$

$e^{\ln \frac{2n!}{n!\,n!} \cdot \frac{2n!}{2^n}}$ Stirling

B. Give an explicit formula for the EGF $C(z) = \sum_{n \geq 0} \frac{C_{2n}}{(2n)!} z^{2n}$

Food for thought after the exam: Give a full derivation via analytic combinatorics.