(0 + 11 + 3 + 9 + 11 + 6 + 11 + 5) 70%

COS 488 Written Exam 1 Spring 2017

There are nine questions on this exam, weighted as indicated at the bottom of this page. There is one question per lecture, numbered corresponding to the lectures, *not in order of difficulty*. If a question seems difficult to you, skip it and come back to it.

Policies. The exam is closed book, though that you are allowed to use a single-page one-sided hand-written cheatsheet. No calculators or other electronic devices are permitted. Give your answers and show your work in the space provided. You will have 80 minutes to complete the test. **This exam will be preprocessed by computer.** If you use pencil (and eraser), write darkly. Fill in circles *completely* when asked. Write all answers inside the designated rectangles. Do not write on the corner marks.

This page. *Print your name and login ID on this page* (now), and write out and sign the Honor Code pledge.

Discussing this exam. As you know, discussing the contents of this exam before solutions have been posted is a serious violation of the Honor Code.

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Maryam Bahrani

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Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	TOTAL
/11	/11	/11	/11	/11	/11	/11	/11	/11	/99

Q1. Analysis of Algorithms. In a particular (fictitious) sorting application with cloud computing, the cost of sorting files of size less than 10^6 is negligible. Otherwise, the cost of comparisons is such that the budget only can cover 10^{12} comparisons. Of the following, which is the largest file that can be sorted within budget, using the standard Quicksort algorithm (with cutoff to a free sort for files of size less than 10^6)?

- 10^9 ○
- 10^{10} ○
- 10^{11} ○
- 10^{12} ●
- 10^{13} ○
- 10^{14} ○

Q2. Recurrences. Match each recurrence with an expression that can make it telescope, by writing the letter to the left of the recurrence in the box to the right of the expression. Find a matching where each letter is used exactly once (and two boxes are left blank).

A $(n+1)a_{n+1} = (n-2)a_{n-1} + n$

B $a_{n+1} = 4a_{n-1} + (n+1)(n+2)$

C $na_n = 4a_{n-1} + (n+1)(n+2)$

D $na_{n+1} = (n+4)a_{n-1} + n + 4$

E $(n+1)a_{n+1} = (n+2)a_{n-1} + n$

$(n+1)(n+2)(n+3)$	D
$n(n+2)(n+4)$	
2^{n+1}	B
$(n+1)!$	
$(n-1)!/4^n$	C
$n(n-1)$	A
$(n+1)(n+2)$	E

αα α

αα α

Q3. Generating functions. Match each sequence with its EGF, by writing the letter to the left of the sequence in the box to the right of the EGF.

		$(e^z - e^{-z})/2$	
		ze^z	C
A	0, −1, 1, −1, 1, −1, . . .	$e^{2z} - e^z$	B
B	0, 1, 3, 7, 15, 31, 63, . . .	$(z^2 + z)e^z$	D
C	0, 1, 2, 3, 4, 5, 6, . . .	$e^{-z} - 1$	A
D	0, 1, 4, 9, 16, 25, 36, . . .	$\dfrac{z}{1-z}$	E
E	0, 1, 2, 6, 24, 120, 720, . . .	$e^{z^2} - 1$	

Q4. Asymptotics. Match each expression with the value that most closely approximates it, by writing the letter to the left of the sequence in the box to the right of the value.

1.001012	C
1.000144	B
1.000023	
1.010050	A
1.001111	
1.000007	E
1.005000	D

A $e^{.01}$

$1+\frac{1}{100}+\frac{1}{2\times 10000}+\cdots$

B $\ln(1000)/\ln(999)$

$\ln(1000^{1/\ln 999}) = \ln(10^{3/\ln 999})$

C $1.001^{694} - 1$

$(1+\frac{1}{1000})^{500} = \sqrt{e} \sim 1.65 \quad (1+\frac{1}{1000})^{194} \sim 1.214$

D $(100 + \ln(100!))/(100\ln(100))$

E $\ln(2.7183)$

$\ln(e+2^{-5})$

$= 1+\ln(1+\frac{2^{-5}}{e})$

$\approx 1+\frac{2^{-5}}{e}$

Q5. Analytic Combinatorics. Indicate the combinatorial class corresponding to each construction (if present), by writing the letter to the left of the sequence in the box to the right of the construction. The correct answer may not use all letters, may leave one or more boxes blank, and/or may use one or more letters more than once.

		$A = Z + A \times A$	E
		invol. $A = SET(CYC_{>1}(Z))$	
A	permutations		
B	binary strings	$A = E + Z \star A$	A
C	cycles		
		$A = SEQ(Z_0 + Z_1)$	B
D	derangements		
E	binary trees	$A = CYC_{>0}(Z)$	C
		$A = A \times (Z_0 + Z_1)$	B
		$A = SEQ(Z)$	A

Q6. Trees. A *binary trie structure* is a binary tree with two types of external nodes (void and nonvoid) with the restriction that void external nodes do not appear in leaves. The number of different binary trie structures with n external nodes is 1, 4, and 17 when n is 2, 3, and 4, respectively. Complete the following derivation to determine an asymptotic approximation to the number of binary trie structures of with n external nodes.

combinatorial construction

$$T = Z_{\blacksquare} + (Z_{\bigcirc} \times T \times T) + 2(Z_{\square} \times Z_{\bigcirc} \times (T - Z_{\blacksquare}))$$

EGF equation

$$T(z) = z + T(z)^2 + 2z\,[T(z) - z]$$

explicit form of EGF

$$T(z) = \frac{1 - 2z - \sqrt{(1-2z)(1-6z)}}{2}.$$

$f(z) = -\frac{1}{2}\sqrt{1-2z}$ $\quad g(z) = \sqrt{1-6z}$ $\quad \rho = \frac{1}{6}$ $\quad \alpha = -\frac{1}{2}$ $\quad -\frac{1}{2}\Gamma(-\frac{1}{2}) = \Gamma(\frac{1}{2}) = \sqrt{\pi} \Rightarrow \Gamma(-\frac{1}{2}) = -2\sqrt{\pi}$

asymptotic approximation

$$[z^N]\,T(z) \sim \frac{-\frac{1}{2}\sqrt{1-\frac{1}{3}}}{-2\sqrt{\pi}} \cdot 6^N \cdot N^{-3/2} = \frac{6^N}{\sqrt{\frac{3}{2}\pi N^3}}.$$

Q7. Permutations. Complete the following derivation to determine the number of permutations of size *n* that have cycles all of even length.

combinatorial construction

Ω = even numbers

$P = SET(CYC_{\Omega}(Z))$

EGF equation

$$P(z) = \exp\left(\sum_{\text{even } k} \frac{z^k}{k}\right) = e^{1 + \frac{z^2}{2} + \frac{z^4}{4} + \cdots}$$

$$= \exp\left(\ln \frac{1}{1-z} - \sum_{\text{odd } k} \frac{z^k}{k}\right) = \frac{e^{-z - \frac{z^3}{3} - \frac{z^5}{5} - \cdots}}{1-z}$$

explicit form of EGF

$$P(z) = \frac{e^{-\sum_{\text{odd } k} \frac{z^k}{k}}}{1-z}$$

exact value of coefficients

$$\left(\frac{(2n)!}{n!2^n}\right)^2$$

Q8. Strings. Determine the OGF for the number of strings not containing the pattern 1011.

combinatorial constructions

S_p: bin. strings w/ no occurance of p = "1011"

T_p: bin. strings terminating in p w/ no other occurrence of p

$S_p + T_p = E + S_p \times \{Z_0 + Z_1\}$

$S_p \times Z_{1011} = T_p \times \{\underbrace{Z_{011}}_{z^3} + \underbrace{E}_{1}\}$

1011
1011 → z^0
1011
1011
1011 → z^3

$C_p(z) = 1 + z^3$

OGF equations

~~$S_p(z) = \frac{C_p(z)}{z^{|p|} + (1-2z)C_p(z)}$~~

~~$S_p(z) = \frac{1+z^3}{z^4 + (1-2z)(1+z^3)}$~~

$S_p(z) + T_p(z) = 1 + 2z S_p(z)$

$S_p(z) \cdot z^4 = T_p(z)\,[z^3 + 1]$

explicit form of OGF

$S_p(z) = \frac{C_p(z)}{z^4 + (1-2z)C_p(z)} \Rightarrow S_p(z) = \frac{1+z^3}{z^4 + (1-2z)(1+z^3)}.$

Q9. Words and Maps. This problem involves enumerating the mappings where every character appears exactly twice or not at all. Such mappings must be of even length. There are 2 such mappings of length 2 (11 and 22) and 36 of them of length 4:

```
1122 1212 1221 2112 2121 2211   1133 1313 1331 3113 3131 3311
1144 1414 1441 4114 4141 4411   2233 2323 2332 3223 3232 3322
2244 2424 2442 4224 4242 4422   3344 3434 3443 4334 4343 4433
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By a direct combinatorial argument, it is not difficult to derive the explicit form

$$C_{2n} = \binom{2n}{n}\frac{(2n)!}{2^n}$$

A. Give an asymptotic estimate (~-approximation) of this quantity.

normal dist. approx w/ k=0: $\binom{2n}{n} \sim \frac{e^{-0^2/n}}{\sqrt{\pi N}} + O(1/N^{3/2}) = \frac{1}{\sqrt{\pi N}} + O(N^{-3/2})$

B. Give an explicit formula for the EGF $C(z) = \sum_{n\geq 0} \frac{C_{2n}}{(2n)!} z^{2n}$

$M = SET(CYC(C))$ ← cayley trees.

Food for thought after the exam: Give a full derivation via analytic combinatorics.