

Q1. Analysis of Algorithms. In a particular (fictitious) sorting application with cloud computing, the cost of sorting files of size less than 10^6 is negligible. Otherwise, the cost of comparisons is such that the budget only can cover 10^{12} comparisons. Of the following, which is the largest file that can be sorted within budget, using the standard Quicksort algorithm (with cutoff to a free sort for files of size less than 10^6)?

10^9

10^{10}

10^{11}

10^{12}

10^{13}

10^{14}

Quicksort: $N \lg N$ compares

Q2. Recurrences. Match each recurrence with an expression that can make it telescope, by writing the letter to the left of the recurrence in the box to the right of the expression. Find a matching where each letter is used exactly once (and two boxes are left blank).

$$(n+1)(n+2)(n+3)$$



$$n(n+2)(n+4)$$



~~A~~ $(n+1)a_{n+1} = (n-2)a_{n-1} + n$

$$2^{n+1}$$



~~B~~ $a_{n+1} = 4a_{n-1} + (n+1)(n+2)$

~~C~~ $na_n = 4a_{n-1} + (n+1)(n+2)$

$$(n+1)!$$



~~D~~ $na_{n+1} = (n+4)a_{n-1} + n+4$

~~E~~ $(n+1)a_{n+1} = (n+2)a_{n-1} + n$

$$(n-1)!/4^n$$



$$n(n-1)$$



$$(n+1)(n+2)$$



Q3. Generating functions. Match each sequence with its EGF, by writing the letter to the left of the sequence in the box to the right of the EGF.

~~A~~ 0, -1, 1, -1, 1, -1, ...

~~B~~ 0, 1, 3, 7, 15, 31, 63, ...

C 0, 1, 2, 3, 4, 5, 6, ...

D 0, 1, 4, 9, 16, 25, 36, ...

~~E~~ 0, 1, 2, 6, 24, 120, 720, ...

~~X~~ $(e^z - e^{-z})/2$

ze^z C

$e^{2z} - e^z$ B

$(z^2 + z)e^z$ D

$e^{-z} - 1$ A

$\frac{z}{1-z}$ E

~~X~~ $e^{z^2} - 1$

$e^z = 1 + z + \frac{z^2}{2!} + \dots$

$\frac{z}{1-z} = \sum_{k=1}^{\infty} \frac{z^k}{k!} \quad 0, 1, 2, 6, \dots$

$e^{(z^2)} = \sum_{k=0}^{\infty} \frac{z^{2k}}{k!} = 1 + z^2 + \frac{z^4}{2!} + \dots$

Q4. Asymptotics. Match each expression with the value that most closely approximates it, by writing the letter to the left of the sequence in the box to the right of the value.

1.001012

1.000144

~~A~~

$e^{.01}$

1.000023

B

$\ln(1000)/\ln(999)$

1.010050

A

C

$1.001^{694} - 1$

D

$(100 + \ln(100!))/(100\ln(100))$

1.001111

~~E~~

$\ln(2.7183)$

1.000007

E

1.005000



Q5. Analytic Combinatorics. Indicate the combinatorial class corresponding to each construction (if present), by writing the letter to the left of the sequence in the box to the right of the construction. The correct answer may not use all letters, may leave one or more boxes blank, and/or may use one or more letters more than once.

		\times $A = Z + (A \times A)$	<input type="checkbox"/>
A	permutations	$A = SET(CYC_{>1}(Z))$	<input type="checkbox" value="D"/>
B	binary strings	$A = E + Z \star A$	<input type="checkbox" value="A"/>
C	cycles	$A = SEQ(Z_0 + Z_1)$	<input type="checkbox" value="B"/>
D	<u>derangements</u> <small>\rightarrow perms w/ no 1-cycle</small>		
E	binary trees	$A = CYC_{>0}(Z)$	<input type="checkbox" value="C"/>
		\times $A = A \times (Z_0 + Z_1)$	<input type="checkbox"/>
		\times $A = SEQ(Z)$ <small>urns</small>	<input type="checkbox"/>

Q6. Trees. A *binary trie structure* is a binary tree with two types of external nodes (void and nonvoid) with the restriction that void external nodes do not appear in leaves. The number of different binary trie structures with n external nodes is 1, 4, and 17 when n is 2, 3, and 4, respectively. Complete the following derivation to determine an asymptotic approximation to the number of binary trie structures of with n external nodes. (void & nonvoid)

■ = non-void external

□ = void external ○ = internal

combinatorial construction

$$T = Z_{\blacksquare} + (Z_{\circ} \times T \times T) + 2(Z_{\square} \times Z_{\circ} \times (T - Z_{\blacksquare}))$$

EGF equation

$$T(z) = z + (T(z))^2 + 2z(T(z) - z)$$

explicit form of EGF

$$0 = (z - 2z^2) + (2z - 1)T(z) + T(z)^2$$

$$T(z) = \frac{(1-2z) - \sqrt{(2z-1)^2 - 4(z-2z^2)}}{2}$$

$$= \frac{(1-2z) - \sqrt{12z^2 - 8z + 1}}{2} = \frac{(1-2z) - \sqrt{(1-8z)(1-6z)}}{2}$$

asymptotic approximation

ignore leading terms + non dominant terms

$$[z^n] \frac{-\frac{1}{2}}{(1-8z)^{\frac{1}{2}}} \sim \frac{-1}{2\Gamma(-\frac{1}{2})} 8^n n^{-3/2}$$

$$\rho = \frac{1}{8}$$

$$\alpha = -\frac{1}{2}$$

$$= \frac{-1}{2\sqrt{\pi}} 8^n n^{-3/2}$$

$$= \frac{1}{2\sqrt{\pi}} 8^n n^{-3/2}$$

$$\Gamma\left(\frac{1}{2}\right) = -\frac{1}{2} \Gamma\left(-\frac{1}{2}\right)$$

$$= \sqrt{\pi}$$

Q7. Permutations. Complete the following derivation to determine the number of permutations of size n that have cycles all of even length.

combinatorial construction

\mathcal{E} = set perms w/ all even length cycles

$$\mathcal{E} = \text{SET} \left(\sum_{k=0}^{\infty} \text{CYC}_{2k}(z) \right)$$

EGF equation

$$E(z) = \exp \left(\sum_{k=0}^{\infty} \frac{z^{2k}}{2k} \right)$$

explicit form of EGF

exact value of coefficients

$$\left(\frac{(2n)!}{n!2^n} \right)^2$$

Q8. Strings. Determine the OGF for the number of strings not containing the pattern 1011. ✓

combinatorial constructions

\mathcal{B} = set bitstrings not containing 1011

\mathcal{T} = set bitstrings with ending in 1011, but not containing the pattern anywhere else

$$\mathcal{B} + \mathcal{T} = \mathcal{E} + \mathcal{B} \times (\mathcal{Z}_0 + \mathcal{Z}_1)$$

$$\mathcal{B} \times \mathcal{Z}_1 \times \mathcal{Z}_0 \times \mathcal{Z}_1 \times \mathcal{Z}_1 = \mathcal{T} \times (\mathcal{E} + \mathcal{Z}_0 \times \mathcal{Z}_1 \times \mathcal{Z}_1)$$

OGF equations

$$B(z) + T(z) = 1 + 2z B(z)$$

$$z^4 B(z) = T(z) (1 + z^3)$$

Autocorrelation poly:

$$c(z) = 1 - z^3$$

explicit form of OGF

$$T(z) = \frac{z^4 B(z)}{1 + z^3}$$

$$\Rightarrow B(z) + \frac{z^4 B(z)}{1 + z^3} = 1 + 2z B(z)$$

$$B(z) (z^4 + (1 - z^3)(1 + z^3)) = 1 + z^3$$

$$B(z) = \frac{1 + z^3}{z^4 + (1 - z^3)(1 + z^3)}$$

Q9. Words and Maps. This problem involves enumerating the mappings where every character appears exactly twice or not at all. Such mappings must be of even length. There are 2 such mappings of length 2 (11 and 22) and 36 of them of length 4:

1122 1212 1221 2112 2121 2211 1133 1313 1331 3113 3131 3311
 1144 1414 1441 4114 4141 4411 2233 2323 2332 3223 3232 3322
 2244 2424 2442 4224 4242 4422 3344 3434 3443 4334 4343 4433

By a direct combinatorial argument, it is not difficult to derive the explicit form

$$C_{2n} = \binom{2n}{n} \frac{(2n)!}{2^n}$$

A. Give an asymptotic estimate (\sim -approximation) of this quantity.

B. Give an explicit formula for the EGF $C(z) = \sum_{n \geq 0} \frac{C_{2n}}{(2n)!} z^{2n}$

$$C(z) = \sum_{n=0}^{\infty} \frac{C_{2n}}{(2n)!} z^{2n} = \sum_{n=0}^{\infty} \binom{2n}{n} \frac{z^{2n}}{2^n}$$

Food for thought after the exam: Give a full derivation via analytic combinatorics.